

2 spring-2 mass system with gravity, easy numbers

a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_f$ in equilibrium \rightarrow solve $A\underline{x} + \underline{f} = \underline{0}$
 at constant values of x_1, x_2 or $A\underline{x} = -\underline{f}$ \leftarrow rref (augment (A, \underline{f}))
 or use inverse:

$$\underline{x} = -A^{-1}\underline{f} = -\frac{1}{6} \begin{bmatrix} -2 & -2 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/6 \\ 7/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 7/6 \end{bmatrix} \sim \begin{bmatrix} 8 \text{ in} \\ 14 \text{ in} \end{bmatrix}$$

explanation: spring 1 is pulled down by $1 \text{ lb} + 1 \text{ lb} = 2 \text{ lb}$ so $k_1 x_1 = 2 \rightarrow x_1 = 2/k_1 = 2/3$
 spring 2 is pulled down by 1 lb so $k_2(x_2 - x_1) = 1 \rightarrow x_2 - x_1 = 1/k_2 = 1/2 \sim 6 \text{ in}$ \uparrow \checkmark

c) now pull down m_2 with force of 2 lbs till reach new equilibrium, repeat with $m_2 = 3$ to find these values:

$$\underline{x} = -A^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -2 & -2 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8/6 \\ 17/6 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 17/6 \end{bmatrix} \sim \begin{bmatrix} 16 \text{ in} \\ 34 \text{ in} \end{bmatrix}$$

if we then release the mass, the system will oscillate about the gravity equilibrium state.

d) we find the "eigenmodes" for these departures from equilibrium:

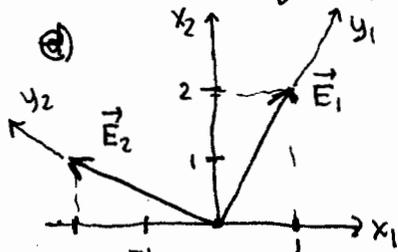
$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda+5)(\lambda+2) - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+1)(\lambda+6) = 0 \rightarrow \lambda = -1, -6 = \lambda_1, \lambda_2$$

$$\lambda = -1: \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 = 1/2 t \\ x_2 = t \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 t \\ t \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \rightarrow \underline{E}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

\leftarrow double

$$\lambda = -6: \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 = -2t \\ x_2 = t \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow \underline{E}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\underline{B} = [\underline{E}_1 \ \underline{E}_2] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad \underline{B}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad \underline{A}_B = \underline{B}^{-1} \underline{A} \underline{B} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \quad \text{diagonalization successful}$$



change of variables illustrated (not requested)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\underline{B}^{-1} \underline{f} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}$$

new eqns of motion: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}$ or $y_1'' = -y_1 + 3/5$
 $y_2'' = -6y_2 - 1/5$

put in "standard form", find homogeneous soln, used method of undetermined coefficients for constants as driving functions:

$$y_1'' + y_1 = 3/5 \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow y_{1h} = c_1 \cos t + c_2 \sin t$$

$$y_2'' + 6y_2 = -1/5 \rightarrow r^2 + 6 = 0 \rightarrow r = \pm \sqrt{6}i \rightarrow y_{2h} = c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t$$

\uparrow constant driving functions \rightarrow trial functions are undetermined constant functions

$$\begin{matrix} y_{1p} = A \rightarrow 0 + A = 3/5 \rightarrow A = 3/5 \\ y_{2p} = B \rightarrow 0 + 6B = -1/5 \rightarrow B = -1/30 \end{matrix} \quad \left. \vphantom{\begin{matrix} y_{1p} = A \\ y_{2p} = B \end{matrix}} \right\} \underline{y}_p = \begin{bmatrix} 3/5 \\ -1/30 \end{bmatrix} \quad \text{this is just gravity equilibrium expressed in the new coords:}$$

$$\underline{x} = \begin{bmatrix} 2/3 \\ 7/6 \end{bmatrix} \rightarrow \underline{y} = \underline{B}^{-1} \underline{x} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 7/6 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/30 \end{bmatrix} \quad \text{they solve } \underline{A}_B \underline{y} + \underline{B}^{-1} \underline{f} = \underline{0} \quad \text{setting } \underline{y}'' = \underline{0} \text{ in eqns}$$

REMARK. Notice the eigenvectors are perpendicular. This is true because the original matrix A is symmetric about its main diagonal: $A_{12} = A_{21}$. Eigenvectors for distinct eigenvalues are always orthogonal (perpendicular) for such "symmetric matrices".

2 spring 2 mass system with gravity, easy numbers (2)

initial values:

$$\underline{x} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos t + c_2 \sin t + 3/5 \\ c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t - 1/30 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 + 3/5 \\ c_3 - 1/30 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 17/6 \end{bmatrix}$$

from part c)

$$\underline{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -c_1 \sin t + c_2 \cos t \\ -\sqrt{6}c_3 \sin t + \sqrt{6}c_4 \cos \sqrt{6}t \end{bmatrix}$$

$$\underline{x}'(0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ \sqrt{6}c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at rest} \rightarrow \begin{matrix} c_2=0 \\ c_4=0 \end{matrix}$$

$$\begin{bmatrix} c_1 + 3/5 \\ c_3 - 1/30 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4/3 \\ 17/6 \end{bmatrix} = \begin{bmatrix} 42/30 \\ 1/30 \end{bmatrix} \rightarrow \begin{matrix} c_1 = 7/5 - 3/5 = 4/5 \\ c_3 = 1/30 + 1/30 = 1/15 \end{matrix}$$

slow mode

fast mode

$$\text{soln: } \underline{x} = \left(\frac{3}{5} + \frac{4}{5} \cos t \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(-\frac{1}{30} + \frac{1}{15} \cos \sqrt{6}t \right) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} + \frac{4}{5} \cos t - \frac{2}{15} \cos \sqrt{6}t \\ \frac{7}{6} + \frac{8}{5} \cos t + \frac{1}{15} \cos \sqrt{6}t \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

g-equilibrium values (y)

g-equilibrium values (x)

slow mode oscillation about g-equilibrium

fast mode oscillation about g-equilibrium

ratio of coeffs: 12 to 1
fast mode contribution small in comparison.

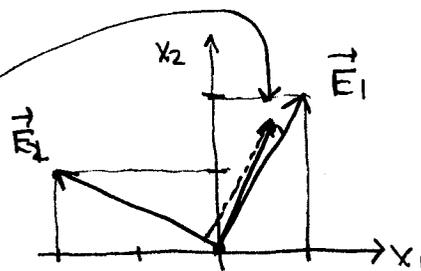
Since frequency ratio $\omega_1/\omega_2 = 1/\sqrt{6}$ is irrational the period ratio $T_1/T_2 = (2\pi/\omega_1)/(2\pi/\omega_2) = \sqrt{6}$ is irrational so one cannot have a common larger period for both $nT_1 = mT_2$ for positive whole numbers n, m since then $\frac{m}{n} = \frac{T_1}{T_2} = \sqrt{6}$ (rationals cannot equal irrationals)
so this solution never repeats in time — it is not periodic.

only the change from g-equilibrium leads to an oscillation. The initial state is pulled downwards from that equilibrium by

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 8/6 - 4/6 \\ 17/6 - 7/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix} = \frac{4}{5} \underline{E}_1 + \frac{1}{5} \underline{E}_2$$

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 1/15 \end{bmatrix}$$

The initial displacement from g-equilibrium is nearly aligned with \underline{E}_1 (the component along \underline{E}_2 is 15 times smaller!) so primarily only the slow mode is "excited" by this pull.



2 components along \underline{E}_1 & \underline{E}_2 just oscillate with different frequencies

