

## 2<sup>nd</sup> order linear nonhomogeneous DE system

$$\underline{x}'' = \underline{A}\underline{x} + \underline{F}, \quad \underline{x}(0) = [0, 1], \quad \underline{x}'(0) = [0, 0], \quad \underline{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix}$$

$$\boxed{\det(\underline{A} - \lambda \underline{I}) = \det \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = (\lambda+5)(\lambda+2) - 4 = \lambda^2 + 7\lambda + 6 = 0}$$

$$\lambda = \frac{-7 \pm \sqrt{49-24}}{2} = -\frac{7 \pm 5}{2} = -1, -6$$

$$\lambda = -1: \quad \begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} L & F \\ 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - \frac{1}{2}x_2 = 0 \quad x_1 = \frac{1}{2}t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kt \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -6: \quad \begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} L & F \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad x_1 = -2t \quad \vec{B}_1 = [1, 2]$$

$$\lambda = -1, -6$$

$$\underline{B} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \underline{B}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad \underline{A}_B = \underline{B}^{-1} \underline{A} \underline{B} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$$

$$\underline{x} = \underline{B} \underline{y}, \quad \underline{y} = \underline{B}^{-1} \underline{x} \rightarrow \underline{y}'' = \underline{A}_B \underline{y} + \underline{B}^{-1} \underline{F}$$

$$\underline{B}^{-1} \underline{F} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \cos 2t \\ \frac{1}{5} \cos 2t \end{bmatrix}$$

$$y_1'' = -y_1 + \frac{2}{5} \cos 2t \quad y_1'' + y_1 = \frac{2}{5} \cos 2t$$

$$y_2'' = -6y_2 + \frac{1}{5} \cos 2t \quad y_2'' + 6y_2 = \frac{1}{5} \cos 2t$$

hom soln:  
(characteristic eqns)

$$r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow y_{1h} = c_1 \cos t + c_2 \sin t$$

$$r^2 + 6 = 0 \rightarrow r = \pm \sqrt{6}i \rightarrow y_{2h} = c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t$$

particular soln:  
(undetermined coefficients)

$$y_{1p} = c_5 \cos 2t + c_6 \sin 2t \rightarrow y_{1p}'' + y_{1p} = -4c_5 \cos 2t - 4c_6 \sin 2t + c_5 \cos 2t + c_6 \sin 2t$$

$$y_{2p} = c_7 \cos 2t + c_8 \sin 2t \rightarrow y_{2p}'' + 6y_{2p} = -4c_7 \cos 2t - 4c_8 \sin 2t + 6c_7 \cos 2t + 6c_8 \sin 2t$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t & -\frac{2}{5} \cos 2t \\ c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t & +\frac{1}{10} \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t - 2(c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t) & +(-\frac{2}{5} - \frac{2}{10}) \cos 2t \\ 2c_1 \cos t + 2c_2 \sin t + c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t & +(-\frac{1}{5} + \frac{1}{10}) \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -c_1 \sin t + c_2 \cos t & +2\sqrt{6}c_3 \sin \sqrt{6}t - 2\sqrt{6}c_4 \cos \sqrt{6}t + 2c_3 \sin 2t \\ -2c_1 \sin t + 2c_2 \cos t & -\sqrt{6}c_3 \sin \sqrt{6}t + \sqrt{6}c_4 \cos \sqrt{6}t + 2c_4 \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 & -2c_3 & -1/3 \\ 2c_1 & +c_3 & -1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & V_3 \\ 2 & 1 & \frac{1}{6} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 8/15 \\ 0 & 1 & 1/10 \end{bmatrix} \quad c_1 = 8/15, \quad c_3 = 1/10$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} c_2 & -2\sqrt{6}c_4 \\ 2c_2 & \sqrt{6}c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow c_2 = 0, \quad c_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8/15 \cos t & -1/5 \cos \sqrt{6}t & -1/3 \cos 2t \\ 16/15 \cos t & +1/10 \cos \sqrt{6}t & -1/6 \cos 2t \end{bmatrix}$$

IVP Solution  
not periodic motion

Because  $\underline{A}$  mixes up  $x_1, x_2$ :  
our trial function form from  
 $\cos 2t$  in  $x_2$  must also hold  
for  $x_1$ :

$$x_{1p} = a_1 \cos 2t + b_1 \sin 2t$$

$$x_{2p} = a_2 \cos 2t + b_2 \sin 2t$$

alternate attack: particular soln by direct backsubstitution in  $\underline{x}$ :

$$x_{1p}'' + 5x_{1p} - 2x_{2p} = -4a_1 \cos 2t - 4b_1 \sin 2t - 2a_2 \cos 2t - 2b_2 \sin 2t = \underbrace{(a_1 - 2a_2)}_0 \cos 2t + \underbrace{(b_1 - 2b_2)}_0 \sin 2t = 0$$

$$x_{2p}'' - 2x_{1p} + 2x_{2p} = -4a_2 \cos 2t - 4b_2 \sin 2t - 2a_1 \cos 2t - 2b_1 \sin 2t = \underbrace{(-2a_1 - 2a_2)}_1 \cos 2t + \underbrace{(-2b_1 - 2b_2)}_0 \sin 2t = \cos 2t$$

$$\rightarrow a_1 = -1/3, \quad a_2 = -1/6$$

$$\rightarrow b_1 = 0, \quad b_2 = 0$$

$$x_{1p} = -1/3 \cos 2t \quad \checkmark$$

$$x_{2p} = -1/6 \cos 2t$$

## 2<sup>nd</sup> order linear homogeneous DE system: resonance

$$\underline{x}'' = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \cos \omega t \end{bmatrix} \quad \text{now repeat for a general frequency } \omega \neq 1, \sqrt{6}$$

decoupled equations:

$$\underline{y}'' = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \underline{y} + \begin{bmatrix} \frac{2}{5} \cos \omega t \\ \frac{1}{5} \cos \omega t \end{bmatrix} \quad \begin{aligned} y_1'' + y_1 &= \frac{2}{5} \cos \omega t \\ y_2'' + 6y_2 &= \frac{1}{5} \cos \omega t \end{aligned}$$

the homogeneous soln  
doesn't change and is  
determined by initial  
conditions

We are only interested in the particular soln which is the response to the driving force.  
Because there is no damping, we only need cosine functions in our trial particular functions;

$$y_{1P} = C_5 \cos \omega t$$

$$y_{1P}'' + y_1 = C_5(-\omega^2 + 1) \cos \omega t = \frac{2}{5} \cos \omega t \rightarrow C_5 = \frac{2}{5(1-\omega^2)}, \omega \neq 1$$

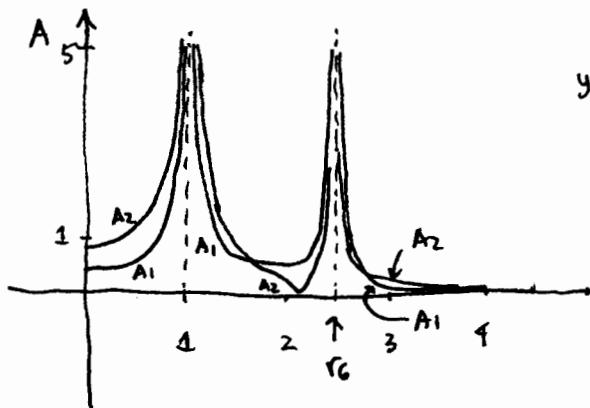
$$y_{2P} = C_7 \cos \omega t$$

$$y_{2P}'' + 6y_{2P} = C_7(-\omega^2 + 6) \cos \omega t = \frac{1}{5} \cos \omega t \rightarrow C_7 = \frac{1}{5(6-\omega^2)}, \omega \neq \sqrt{6}$$

$$\begin{bmatrix} x_{1P} \\ x_{2P} \end{bmatrix} = B \begin{bmatrix} y_{1P} \\ y_{2P} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{5(1-\omega^2)} \\ \frac{1}{5(6-\omega^2)} \end{bmatrix} \cos \omega t = \begin{bmatrix} \frac{2}{5}(1-\omega^2 - \frac{1}{6}\omega^2) \\ \frac{1}{5}(1-\omega^2 + \frac{1}{6}\omega^2) \end{bmatrix} \cos \omega t = \frac{\cos \omega t}{(1-\omega^2)(6-\omega^2)} \begin{bmatrix} 2 \\ 5-6\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} \pm A_1 \cos \omega t \\ \pm A_2 \cos \omega t \end{bmatrix}$$

$$A_1 = \frac{2}{|(1-\omega^2)(6-\omega^2)|}, \quad A_2 = \frac{5-\omega^2}{|(1-\omega^2)(6-\omega^2)|}$$



you need to see this in a technology plot!

If some small amount of damping were added to the system these would become resonance peaks near the natural frequencies.