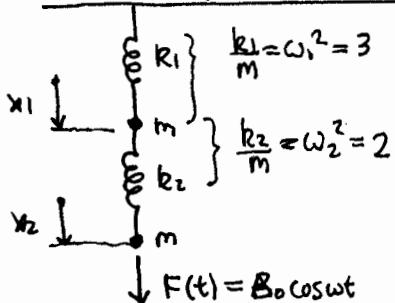


2 mass - 2 spring system : resonance



$$\frac{k_1}{m} = \omega_1^2 = 3 \quad \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]'' = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + \underbrace{\begin{bmatrix} 0 \\ B_0 \cos wt \end{bmatrix}}_{F(t)}$$

A:

$$\lambda = -1, -6 \quad \underline{B} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \underline{B}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\underline{B}^{-1} \begin{bmatrix} 0 \\ B_0 \cos wt \end{bmatrix} = \begin{bmatrix} \frac{2B_0}{5} \cos wt \\ \frac{B_0 \cos wt}{5} \end{bmatrix}$$

separate spring mass system frequencies are:

$$\omega_1 = \sqrt{3} \approx 1.732$$

$$\omega_2 = \sqrt{2} \approx 1.414$$

coupled system:

$$\omega_+ = 1$$

$$\omega_- = \sqrt{6} \approx 2.449$$

undriven system oscillates at frequencies $\omega \pm$

$$y_1'' = -y_1 + \frac{2B_0}{5} \cos wt \rightarrow y_1'' + y_1 = \frac{2B_0}{5} \cos wt$$

$$y_2'' = -6y_2 + \frac{B_0}{5} \cos wt \rightarrow y_2'' + 6y_2 = \frac{B_0}{5} \cos wt$$

$$y_{1h} = c_1 \cos \omega t + c_2 \sin \omega t$$

$$y_{2h} = c_3 \cos \omega_- t + c_4 \sin \omega_- t$$

If $\omega \neq \omega \pm$:

$$y_{1p} = c_5 \cos \omega t + c_6 \sin \omega t$$

$$y_{1p}'' = -\omega^2 c_5 \cos \omega t - \omega^2 c_6 \sin \omega t$$

$$y_{1p}'' + y_1 = \frac{(1-\omega^2)c_5 \cos \omega t + (1-\omega^2)c_6 \sin \omega t}{\omega^2} = \frac{2}{5} B_0 \cos \omega t$$

$$= \frac{2}{5} B_0 \rightarrow c_5 = \frac{2B_0}{5(1-\omega^2)}$$

$$y_{1p} = \frac{2B_0}{5(1-\omega^2)} \cos \omega t$$

$$y_{2p} = c_7 \cos \omega t + c_8 \sin \omega t$$

$$y_{2p}'' = -\omega^2 c_7 \cos \omega t - \omega^2 c_8 \sin \omega t$$

$$y_{2p}'' + 6y_{2p} = \frac{(6-\omega^2)c_7 \cos \omega t + (6-\omega^2)c_8 \sin \omega t}{\omega^2} = \frac{1}{5} B_0 \cos \omega t$$

$$c_7 = \frac{B_0}{5(6-\omega^2)}$$

$$y_{2p} = \frac{B_0}{5(6-\omega^2)} \cos \omega t$$

$$\begin{bmatrix} x_{1p} \\ x_{2p} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2B_0}{5(1-\omega^2)} \cos \omega t \\ \frac{B_0}{5(6-\omega^2)} \cos \omega t \end{bmatrix}$$

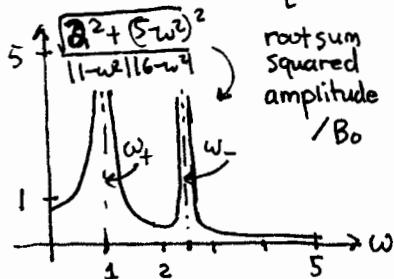
$$= \frac{B_0 \cos \omega t}{(1-\omega^2)(6-\omega^2)} \begin{bmatrix} 2 \\ 5-\omega^2 \end{bmatrix}$$

$$= B_0 \cos \omega t \left(\underbrace{\frac{2}{5(1-\omega^2)} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{relative strength of each mode excited by the driving force}} + \underbrace{\frac{1}{5(6-\omega^2)} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{relative strength of each mode excited by the driving force}} \right)$$

for ω near $\omega \pm = 1, \sqrt{6}$
get large amplitude response

[a real system always has some damping
(which replaces the vertical asymptotes with finite resonance peaks)]

these give ratio of amplitudes for the oscillations of x_1 and x_2



Without damping the homogeneous soln is not a transient (does not decay away),
so the general motion is a linear combination of 3 frequencies: 2 free modes at frequencies ω_+ and ω_- and the response mode at the driving frequency ω .

Initial conditions fix the 2 free modes.

To take damping into account we are forced to use reduction of order to reduce this to a first order system for the initial condition or state vector:

(introducing $x_3 = x_1'$, $x_4 = x_2'$)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}'' = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' + \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 2 & -1 & 0 \\ 2 & -2 & 0 & -1 \end{bmatrix}}_{4 \times 4 \text{ matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

4x4 matrix \rightarrow 4 eigenvalues = 2 complex conjugate pairs:
each associated with a damped oscillation mode

2nd order linear nonhomogeneous DE system

$$\underline{x}'' = \underline{A}\underline{x} + \underline{F}, \quad \underline{x}(0) = [0, 1], \quad \underline{x}'(0) = [0, 0], \quad \underline{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix}$$

$$0 = \det(\underline{A} - \lambda \underline{I}) = \det \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = (\lambda+5)(\lambda+2) - 4 = \lambda^2 + 7\lambda + 6 = 0$$

$$\lambda = \frac{-7 \pm \sqrt{49-24}}{2} = -\frac{7 \pm 5}{2} = -1, -6$$

$\lambda = -1: \begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} L & F \\ 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - \frac{1}{2}x_2 = 0 \quad x_1 = \frac{1}{2}t$

$x_2 = t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -6: \begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} L & F \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0 \quad x_1 = -2t$

$x_2 = t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\underline{B} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \underline{B}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad \underline{A}_B = \underline{B}^{-1} \underline{A} \underline{B} = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$

$\underline{B}^{-1} \underline{F} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \cos 2t \\ \frac{1}{5} \cos 2t \end{bmatrix}$

$$\underline{x} = \underline{B} \underline{y}, \quad \underline{y} = \underline{B}^{-1} \underline{x} \rightarrow \underline{y}'' = \underline{A}_B \underline{y} + \underline{B}^{-1} \underline{F}$$

$$\underline{y}_1'' = -y_1 + \frac{2}{5} \cos 2t \quad y_1'' + y_1 = \frac{2}{5} \cos 2t$$

$$y_2'' = -6y_2 + \frac{1}{5} \cos 2t \quad y_2'' + 6y_2 = \frac{1}{5} \cos 2t$$

hom soln:

(characteristic eqns)

$$r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow y_{1h} = c_1 \cos t + c_2 \sin t$$

$$r^2 + 6 = 0 \rightarrow r = \pm \sqrt{6}i \rightarrow y_{2h} = c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t$$

particular soln: $y_{1p} = c_5 \cos 2t + c_6 \sin 2t \rightarrow y_{1p}'' + y_{1p} = -4c_5 \cos 2t - 4c_6 \sin 2t + c_5 \cos 2t + c_6 \sin 2t$

$y_{2p} = c_7 \cos 2t + c_8 \sin 2t \rightarrow y_{2p}'' + 6y_{2p} = -4c_7 \cos 2t - 4c_8 \sin 2t + 6c_7 \cos 2t + 6c_8 \sin 2t$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t & -\frac{2}{5} \cos 2t \\ c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t & +\frac{1}{10} \cos 2t \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t - 2(c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t) & +(-\frac{2}{5} - \frac{2}{10}) \cos 2t \\ 2c_1 \cos t + 2c_2 \sin t + c_3 \cos \sqrt{6}t + c_4 \sin \sqrt{6}t & +(-\frac{4}{5} + \frac{1}{10}) \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -c_1 \sin t + c_2 \cos t & +2\sqrt{6}c_3 \sin \sqrt{6}t - 2\sqrt{6}c_4 \cos \sqrt{6}t + 2c_3 \sin 2t \\ -2c_1 \sin t + 2c_2 \cos t & -\sqrt{6}c_3 \sin \sqrt{6}t + \sqrt{6}c_4 \cos \sqrt{6}t + 2c_4 \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 & -2c_3 - 1/3 \\ 2c_1 & +c_3 - 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & \sqrt{3} \\ 2 & 1 & \sqrt{6}/6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 8/15 \\ 0 & 1 & 1/10 \end{bmatrix} \quad c_1 = 8/15, \quad c_3 = 1/10$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} c_2 & -2\sqrt{6}c_4 \\ 2c_2 & \sqrt{6}c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow c_2 = 0, \quad c_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8/15 \cos t & -1/5 \cos \sqrt{6}t & -1/3 \cos 2t \\ 16/15 \cos t & +1/10 \cos \sqrt{6}t & -1/6 \cos 2t \end{bmatrix}$$

alternate attack: particular soln by direct back-substitution in \underline{x} :

$$x_{1p}'' + 5x_{1p} - 2x_{2p} = -4a_1 \cos 2t - 4b_1 \sin 2t - 2a_2 \cos 2t - 2b_2 \sin 2t = \underbrace{(a_1 - 2a_2)}_0 \cos 2t + \underbrace{(b_1 - 2b_2)}_0 \sin 2t = 0$$

$$x_{2p}'' - 2x_{1p} + 2x_{2p} = -4a_2 \cos 2t - 4b_2 \sin 2t - 2a_1 \cos 2t - 2b_1 \sin 2t = \underbrace{(-2a_1 - 2a_2)}_1 \cos 2t + \underbrace{(-2b_1 - 2b_2)}_0 \sin 2t = \cos 2t$$

$$\rightarrow a_1 = -1/3, \quad a_2 = -1/6$$

$$\rightarrow b_1 = 0, \quad b_2 = 0$$

$$x_{1p} = -1/3 \cos 2t \quad \checkmark$$

$$x_{2p} = -1/6 \cos 2t$$

Because \underline{A} mixes up x_1, x_2 : our trial function form from $\cos 2t$ in x_2 must also hold for x_1 :

$$x_{1p} = a_1 \cos 2t + b_1 \sin 2t$$

$$x_{2p} = a_2 \cos 2t + b_2 \sin 2t$$