Coupled mass spring systems: the rest of the story

Final Topic: $\boxed{\text{reduction of order}}$ to handle damping in coupled oscillator systems

The textbook does not finish the story we have been building towards all semester. Yes, we can calculate the natural frequencies of these model oscillating systems, but resonance also requires understanding how damping affects these systems.

Our $\boxed{\text{eigenvector decoupling approach}}$ is very powerful for systems of DEs which can be put into the multidimensional matrix first order system $\boxed{x' = Ax + F\,,\ x = \langle x_1, \ldots, x_n \rangle}$

where A is a constant $n \times n$ coefficient matrix, as long as it is diagonalizable either over the real or complex numbers. The nondiagonal case can be handled with an approach similar to the case of repeated roots in chapter 5, which leads to arbitrary polynomial coeffients of exponential function solutions. For defective matrices (repeated roots of the characteristic eqn) which are not diagonalizable, polynomial functions get mixed up with eigenvector components to form "generalized eigenvectors" and Jordan canonical form replaces diagonalization, discussed in section 7.6. This is much less common and better left to a higher level course in DEs or linear algebra.

$\boxed{\text{Reduction of order}}$ allows us to reduce any higher order system of DEs to an equivalent first order system in more variables. We use this technique on higher order linear constant coefficient systems of DEs.

Coupled mass spring systems: the rest of the story

Reduction of order The idea is simple and relies on the idea of initial data for a DE system. For any DE system which can be solved for the highest derivative of each variable appearing in the system, that highest derivative is then determined by the variable and all of its derivatives up to one less than that derivative (for all the variables). This initial data information determines a unique soln since in theory one can iterate the derivative of the DEs to evaluate all higher derivatives of the variables of the system to obtain local solns in principle through Taylor series representations.

The **state vector** or equivalently **initial condition vector** lists all of these initial data variables, and its value at any particular value of the independent variable (t for us!) are a set of initial data we can impose on the general soln.

The derivative of the state vector then includes all the highest derivatives of all the variables, which in turn can be expressed in terms of those state vector variables alone through the DEs of the system.

This idea is best understood with an explicit example.

There are a lot of words here since the textbook does not discuss this, but fortunately the examples make the idea clear!

7.5d) Coupled mass spring systems: the rest of the story

Example 1 $\quad x'' + 3x' + 2x = \cos 2t$ or $x'' = -2x - 3x' + \cos 2t$

Introduce the state vector of the single variable x and its derivative

$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$ By numbering its components we have introduced a new variable $x_2 = x'$ (the velocity if x is position).

We rewrite the DE as $x_2' = x'' = -2x_1 - 3x_2 + \cos 2t$.

Next consider

$$\vec{X}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 + \cos 2t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{X}} + \underbrace{\begin{bmatrix} 0 \\ \cos 2t \end{bmatrix}}_{F}$$

$$= A \quad \vec{X} \ + \ F$$

Success! We now have a first order linear system in 2 variables. The eigenvalues and eigenvectors of A help us write down the general homogeneous soln BUT we already know how to solve this using trial exponential solns.

$$x = e^{rt} \to \underbrace{(r^2 + 3r + 2)}_{=0} e^{rt} = 0$$

$$\to r = -1, -2 \to e^{rt} = e^{-t}, e^{-2t}$$

$$\longrightarrow x = c_1 e^{-t} + c_2 e^{-2t} \quad \text{gen hom soln}$$

so $\quad x' = -c_1 e^{-t} - 2c_2 e^{-2t}$

and $\quad \vec{X} = \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-2t} \\ -c_1 e^{-t} - 2c_2 e^{-2t} \end{bmatrix} = c_1 e^{-t} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b_1} + c_2 e^{-2t} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{b_2}$

But as a solution of:

$X' = AX$ we know the soln is of the form $\to \vec{X} = c_1 e^{r_1 t} b_1 + c_2 e^{r_2 t} b_2$

so comparing them we can identify the eigen values and eigenvectors used here.

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \xrightarrow[\text{Maple}]{} \begin{cases} B = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix} = \langle -b_1 \mid -\tfrac{1}{2} b_2 \rangle \\ \lambda = -1, -2 \end{cases}$$

rescaled to make first component 1 instead of second component!

The eigenvalues (roots of the characteristic eqn of A) are exactly the exponential rate factors which are the roots of the characteristic equation for the scalar DE!

<u>Example 2</u> Now let's try a 2 variable DE system that we know how to solve so we can compare with the resulting reduced system.

$$x_1'' + x_1' + 2x_1 - x_2 = 0 \qquad \text{or} \qquad x_1'' = -2x_1 + x_2 - x_1'$$
$$x_2'' + x_2' - x_1 + 2x_2 = \cos t \qquad\qquad x_2'' = x_1 - 2x_2 - x_2' + \cos 2t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix}$$

$$x'' = \underbrace{A x - I}_{\text{2 coefficient matrices}} x' + F$$

BUT always simultaneously diagonalizable

If $A \longrightarrow A_B = B^{-1} A B$ is diagonal then

$cI \longrightarrow cI_B = B^{-1} cIB = c\,B^{-1} I B = cB^{-1}B = cI$ unchanged!

so multiples of the identity can be handled to introduce a very special damping term coefficient where we dont need reduction of order, but which can be used for comparison.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \xrightarrow[\text{Maple}]{} B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rightarrow B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\lambda = -1, -3 \qquad A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$B^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$x = By$, $y = B^{-1}x$

$$B^{-1}F = \frac{1}{2}\cos 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We get decoupled DEs:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \frac{1}{2}\cos 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -y_1 - y_1' + \frac{1}{2}\cos 2t \\ -3y_2 - y_2' + \frac{1}{2}\cos 2t \end{bmatrix}$$

$y_1'' + y_1' + y_1 = \frac{1}{2}\cos 2t$: $y_{1h} \sim e^{rt}$: $r^2 + r + 1 = 0 \rightarrow r = \frac{-1 \pm i\sqrt{3}}{2} \rightarrow e^{rt} = e^{-t/2} e^{\pm i\sqrt{3}/2\, t}$

$y_2'' + y_2' + 3y_2 = \frac{1}{2}\cos 2t$: $y_{2h} \sim e^{rt}$: $r^2 + r + 3 = 0 \rightarrow r = \underbrace{\frac{-1 \pm i\sqrt{11}}{2}} \rightarrow e^{rt} = e^{-t/2} e^{\pm i\sqrt{11}/2\, t}$

define: $\lambda_{1\pm}$, $\lambda_{2\pm}$

$y_{1h} = e^{-t/2}\left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t\right)$

$y_{2h} = e^{-t/2}\left(c_3 \cos \frac{\sqrt{11}}{2} t + c_4 \sin \frac{\sqrt{11}}{2} t\right)$

$y_{1p} = c_5 \cos 2t + c_6 \sin 2t \longrightarrow -4(c_5 \cos 2t + c_6 \sin 2t) + (-2c_5 \sin 2t + 2c_6 \cos 2t) + (c_5 \cos 2t + c_6 \sin 2t) = \frac{1}{2}\cos 2t$

$y_{2p} = c_7 \cos 2t + c_8 \sin 2t \rightarrow -4(c_7 \cos 2t + c_8 \sin 2t) + (-2c_7 \sin 2t + 2c_8 \cos 2t) + 3(c_7 \cos 2t + c_8 \sin 2t) = \frac{1}{2}\cos 2t$

$$\begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_5 \\ c_6 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} c_7 \\ c_8 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_7 \\ c_8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$x_p = By_p = \frac{1}{65}\begin{bmatrix} -\cos 2t - 8\sin 2t \\ -14\cos 2t + 18 \sin 2t \end{bmatrix}$$

maple!

Yeek! let's take a detour!

Coupled mass spring systems, the rest of the story

First let's summarize. Our general soln is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-\frac{t}{2}}\left(c_1 \cos \tfrac{\sqrt{3}}{2}t + c_2 \sin \tfrac{\sqrt{3}}{2}t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-\frac{t}{2}}\left(c_3 \cos \tfrac{\sqrt{11}}{2}t + c_4 \sin \tfrac{\sqrt{11}}{2}t\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \underbrace{\frac{1}{65}\begin{bmatrix} -\cos 2t - 8\sin 2t \\ -14\cos 2t + 18 \sin 2t \end{bmatrix}}$$

home office distraction ☹

$$x_p = \frac{1}{\sqrt{65}}\begin{bmatrix} \cos(2t + \pi - \arctan 8) \\ 2\sqrt{2}\cos(2t - \pi + \arctan\frac{9}{7}) \end{bmatrix}$$

$$x_p' = \frac{1}{65}\begin{bmatrix} 2\sin 2t - 16\cos 2t \\ 28 \sin 2t + 36 \cos 2t \end{bmatrix}$$

$$x_p(0) = \frac{1}{65}\begin{bmatrix} -1 \\ -14 \end{bmatrix} \qquad x_p'(0) = \frac{4}{65}\begin{bmatrix} -4 \\ 9 \end{bmatrix} \qquad \leftarrow$$

Initial conditions fit c_1, c_2, c_3, c_4 to adjust the initial conditions of the response function to fit the desired initial data.

If we remove the driving term, we merely fit them directly to the initial conditions. [cosine terms excited by nonzero positions, sine terms by nonzero velocities]

The important point is that the details are many BUT not conceptually difficult.

The introduction of this simple damping slowed down the two natural frequencies from $1, \sqrt{3} \approx 1.73$

to $\frac{\sqrt{3}}{2}, \frac{\sqrt{11}}{2} \approx 0.87, 1.66$ elongating the natural periods

from $2\pi \approx 6.28$, $\frac{2\pi}{\sqrt{6}} \approx 2.57$

to $7.26, 3.79$.

If we "release from rest" the 2 masses (zero initial velocities) only the cosine terms contribute (undriven case).

If we only "kick" the 2 masses at equilibrium (zero initial positions), only the sine terms contribute.

"Mixed" initial conditions generate by cosine & sine terms and hence phase shifts.

Now let's do reduction of order.

$$\underline{X} = \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

We have two new "velocity" variables $x_3 = x_1'$ and $x_4 = x_2'$.

Rewrite the DEs using them:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - I \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + F$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}' = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - I \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + F$$

Evaluate:

$$\underline{X}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -2x_1 + x_2 - x_3 \\ x_1 - 2x_2 - x_4 + \cos 2t \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -2 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{array}\right]}_{\mathcal{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\underline{X}} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 2t \end{bmatrix}$$

$$\underline{X}' = \mathcal{A}\,\underline{X} + \underline{F}$$

$$\mathcal{A} = \underbrace{\left[\begin{array}{c|c} 0 & I \\ \hline A & -I \end{array}\right]}$$

} definition of new vars!

} original DE system!

"block matrix"

diagonalize

$$\mathcal{A} \longrightarrow \mathcal{A}_{\mathcal{B}} = \mathcal{B}^{-1} \mathcal{A} \mathcal{B}$$

soln will be:

$$\underline{X} = c_1 e^{\lambda_1 t} b_1 + c_2 e^{\lambda_2 t} b_2 + c_3 e^{\lambda_3 t} b_3 + c_4 e^{\lambda_4 t} b_4 + \underline{X}_p$$

where the eigenvectors and eigenvalues are:

$$\lambda = \begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ -\frac{1}{2} + \frac{i\sqrt{11}}{2} \\ -\frac{1}{2} - \frac{i\sqrt{11}}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} \lambda_{1+} \\ \lambda_{1-} \\ \lambda_{2+} \\ \lambda_{2-} \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & \frac{1}{6} + \frac{i\sqrt{11}}{6} & \frac{1}{6} - \frac{i\sqrt{11}}{6} \\ -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{6} - \frac{i\sqrt{11}}{6} & -\frac{1}{6} + \frac{i\sqrt{11}}{6} \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

divide each column by its second entry

(2x2 eigenvectors have second entry = 1)

$$\text{or } \mathcal{B}_2 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{11}}{2} & \frac{1}{2} - \frac{i\sqrt{11}}{2} \\ -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{1}{2} - \frac{i\sqrt{11}}{2} & \frac{1}{2} + \frac{i\sqrt{11}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_1 & b_2 & b_2 \\ \lambda_{1+} b_1 & \lambda_{1-} b_1 & \lambda_{2+} b_2 & \lambda_{2-} b_2 \end{bmatrix}$$

the eigenvalue appears from the derivative!

To compare to the 2X2 soln we got bogged down in, those real solns came from a complex basis of the soln space consisting of two complex conjugate vector pairs: $e^{\lambda_1 \pm t} b_1,\ e^{\lambda_2 \pm t} b_2$ (homogeneous part!)

The corresponding statevectors are: $e^{\lambda_1 \pm t}\begin{bmatrix} b_1 \\ \lambda_1 \pm b_1 \end{bmatrix},\ e^{\lambda_2 \pm t}\begin{bmatrix} b_2 \\ \lambda_2 \pm b_2 \end{bmatrix}$

The statevector of the general homogeneous soln consists of an arbitrary linear combination of the real and imaginary parts of these two pairs and if we had the patience, we could verify this by hand, but instead we can just examine Maple's direct soln and recognize the above form.

If we consider generic damping with a coefficient matrix not proportional to the 2x2 identity matrix, the more complicated eigenvalues and eigenvectors will characterize the two 2x2 matrices acting together. This is how Maple comes up with the solns. We can use Maple to explore the details, letting Maple do the heavy lifting.

So let's back off and just consider the undamped undriven initial value problem.

 Coupled mass spring system: the rest of the story

Summary of what we need:

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \to B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A^{-1} = \frac{1}{4-1}\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda = -1, -3$$
$$\omega = 1, \sqrt{3} \approx 1.73$$

Suppose we release the 2 masses from rest at a new equilibrium which corresponds to pulling the second mass to the right with a unit force per unit mass in the sense:

$$x'' = Ax + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \leftarrow \text{eqns already divided thru by their masses}$$

A new equilibrium satisfies $x = x_0$: $0 = Ax_0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to x_0 = -A^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

So choose: $\underbrace{x_1(0) = 1/3, \; x_2(0) = 2/3}_{\text{at new equilibrium}}, \; \underbrace{x_1'(0) = 0, \; x_2'(0) = 0}_{\text{at rest}}.$

The undamped general soln just has oscillations along each eigenvector at the natural frequencies

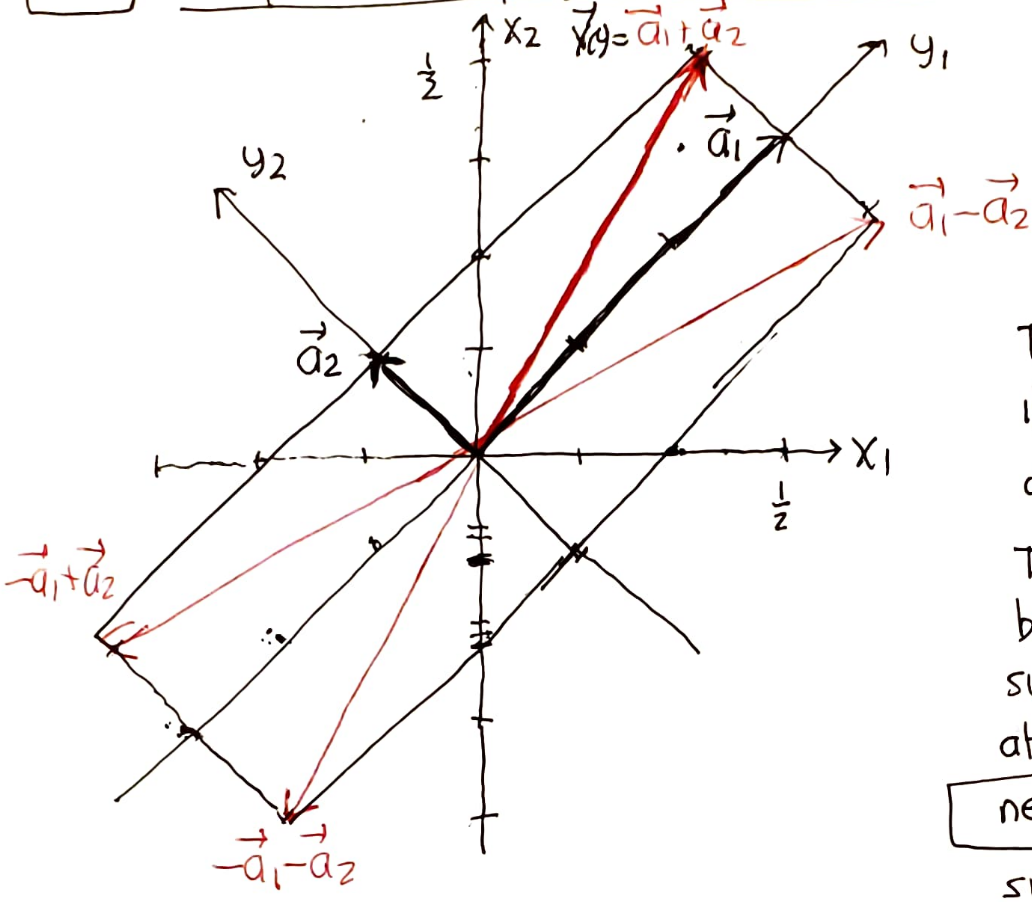
$$x = (c_1 \cos t + c_2 \sin t)\, b_1 + (c_3 \cos\sqrt{3}t + c_4 \sin\sqrt{3}t)\, b_2$$
$$x' = (-c_1 \sin t + c_2 \cos t)\, b_1 + (-\sqrt{3}c_3 \sin\sqrt{3}t + \sqrt{3}c_4 \cos\sqrt{3}t)\, b_2$$
$$x(0) = c_1 b_1 + c_3 b_2 = B\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$x'(0) = c_2 b_1 + \sqrt{3}c_4 b_2 = B\begin{bmatrix} c_2 \\ \sqrt{3}c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} c_2 \\ \sqrt{3}c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[initial positions determine cosine coefficients
initial velocities determine sine coefficients]

$$x = \frac{1}{2}\cos t\, b_1 + \frac{1}{6}\cos\sqrt{3}t\, b_2 = \cos t\,(\underbrace{\tfrac{1}{2}b_1}_{\equiv a_1}) + \cos\sqrt{3}t\,(\underbrace{\tfrac{1}{6}b_2}_{\equiv a_2})$$

The coefficients a_1, a_2 determine a bounding box containing the soln curve, with $x(0) = a_1 + a_2$

7.5d Coupled mass spring system: the rest of the story



The initial position is at one corner of this box.

The solution oscillates between opposite sides of this box at different frequencies $\boxed{\text{never repeating itself}}$ since there is no common period of the motion.

Thus as $t \to \infty$ it will fill in the entire rectangle.

[see Maple].

A similar box is easily calculated if we start out at equilibrium (the origin) with only nonzero initial velocities which instead only excite the sine terms.

For general initial conditions we start out somewhere in the box. The phase shifted form of the decoupled variables determines the box from the amplitudes:

$$x = A_1 \cos(t-\delta_1)\, b_1 + A_2 \cos(\sqrt{3}t-\delta_2)\, b_2$$
$$= \cos(t-\delta_1)\,(\underbrace{A_1 b_1}_{\equiv a_1}) + \cos(\sqrt{3}t-\delta_2)\, \underbrace{A_2 b_2}_{\equiv a_2}$$

Damping causes the boundary of the box to shrink with time since the amplitudes acquire decaying exponential factors.

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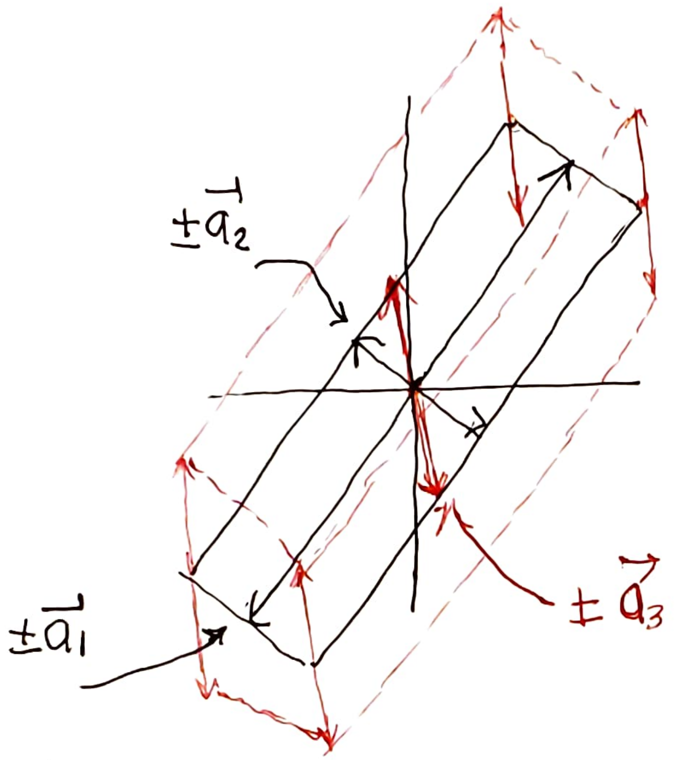
Driving the undamped system adds a third oscillation to the solution which then expands the box boundaries



driving function at frequency ω and some phase shift

$$x = \underbrace{\cos(t-\delta_1)\,\vec{a}_1 + \cos(\sqrt{3}t-\delta_2)\,\vec{a}_2}_{\text{2 eigenmodes}} + \underbrace{\cos(\omega t-\delta_3)\,\vec{a}_3(\omega)}_{\text{response function}}$$

The soln can wander anywhere in the extended boundary region.

The magnitude $|\vec{a}_3(\omega)|$ represents the overall amplitude of the response function and as you might suspect gets large at the natural frequencies where resonance occurs.