

7.5b

Driven coupled mass spring systems

①

The coupled mass spring systems (no damping) have "eigenmodes" with natural frequencies at which they like to oscillate with all displacements either in phase or 180° out of phase in a given such mode. If we drive these systems with oscillating forcing functions at a fixed frequency, we expect a big response near the system's natural frequencies exactly as in the single mass spring system.

Recall $x'' + \omega_0^2 x = f_0 \cos \omega t$, $\omega \neq \omega_0$, $f_0 > 0$

$$x = \underbrace{c_1 \cos \omega_0 t + c_2 \sin \omega_0 t}_{\text{homogeneous soln } x_h \text{ "natural behavior"}} + \underbrace{c_3 \cos \omega t + c_4 \sin \omega t}_{\text{response function } x_p \text{ coeffs determined by method of undetermined coeffs}}$$

$$\omega^2 [x_p = c_3 \cos \omega t + c_4 \sin \omega t]$$

$$1 [x_p'' = -\omega^2 (c_3 \cos \omega t + c_4 \sin \omega t)]$$

$$x_p'' + \omega_0^2 x_p = (\omega_0^2 - \omega^2)(c_3 \cos \omega t + c_4 \sin \omega t) = f_0 \cos \omega t$$

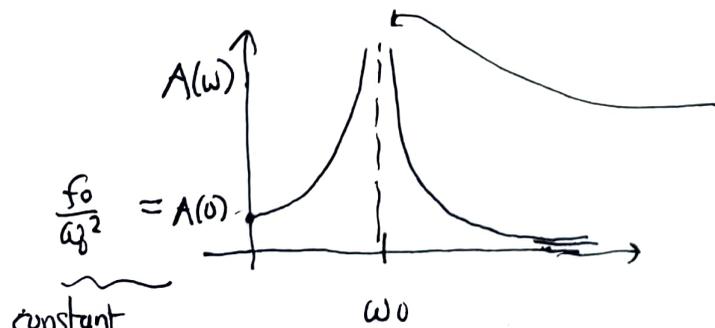
$$(\omega_0^2 - \omega^2) c_3 = f_0 \quad (\omega_0^2 - \omega^2) c_4 = 0$$

$$c_3 = \frac{f_0}{\omega_0^2 - \omega^2} \quad c_4 = 0$$

$$x_p = \frac{f_0}{\omega_0^2 - \omega^2} \cos \omega t$$

$$A(\omega) = \frac{f_0}{|\omega_0^2 - \omega^2|}$$

$\frac{f_0}{\omega_0^2 - \omega^2} > 0, \omega < \omega_0$ in phase
 $< 0, \omega > \omega_0$ 180° out of phase



damping "caps off" this vertical asymptote with a maximum value "resonance peak"

constant driving force

system: can keep up in phase can't keep up out of phase

with driving oscillation.

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Driven coupled mass spring systems

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When we drive a coupled mass spring system with driving forces of a fixed frequency, we consider the nonhomogeneous system:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \leftarrow \quad f_0 = f_{i0} \cos \omega t, \quad \omega \neq \omega_i$$

undriven system:
natural frequencies
 $\omega_i = \sqrt{-\lambda_i}$

$$\cos \omega t \begin{bmatrix} f_{10} \\ f_{20} \end{bmatrix}$$

$$x'' = Ax + f \quad \leftarrow \quad x = By$$

$$\downarrow$$

$$B^{-1} [By]'' = B^{-1}A (By) + B^{-1}f$$

$$y'' = A_B y + \underbrace{B^{-1}f}_{\text{new components of vector } f}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \cos \omega t \begin{bmatrix} (B^{-1}f_0)_1 \\ (B^{-1}f_0)_2 \end{bmatrix}$$

$$y_1'' = \lambda_1 y_1 + (B^{-1}f_0)_1 \cos \omega t$$

$$y_2'' = \lambda_2 y_2 + (B^{-1}f_0)_2 \cos \omega t$$

} uncoupled DEs, solve as in chapter 5

$$y_1'' - \lambda_1 y_1 = (B^{-1}f_0)_1 \cos \omega t$$

$$y_2'' - \lambda_2 y_2 = (B^{-1}f_0)_2 \cos \omega t$$

$$\uparrow$$

$$+ \omega_1^2$$

$$+ \omega_2^2$$

$\rightarrow y_{pi}$ = oscillations at frequency ω ,
coeffs determined by method of
undetermined coeffs

$\rightarrow y_{hi}$ = oscillations at
frequency ω_i

"response functions"

Now let's get concrete with actual numbers.

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -5/2 & 3/2 \\ 3/2 & -5/2 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 50 \cos 3t \end{bmatrix}}_f \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda = -1, -4$

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_B = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$B^{-1} f_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 50 \end{bmatrix} = \begin{bmatrix} 25 \\ 25 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 25 \cos 3t \\ 25 \cos 3t \end{bmatrix} = \begin{bmatrix} -y_1 + 25 \cos 3t \\ -4y_2 + 25 \cos 3t \end{bmatrix}$$

$$y_1'' + y_1 = 25 \cos 3t \rightarrow y_{1h} = c_1 \cos t + c_2 \sin t \quad y_{1p} = c_5 \cos 3t \quad (\text{Sine terms unnecessary})$$

$$y_2'' + 4y_2 = 25 \cos 3t \rightarrow y_{2h} = c_3 \cos 2t + c_4 \sin 2t \quad y_{2p} = c_6 \cos 3t$$

$$y_{1p}'' + y_{1p} = (-9 + 1) c_5 \cos 3t = 25 \cos 3t \rightarrow c_5 = \frac{25}{-8} = -25/8$$

$$y_{2p}'' + 4y_{2p} = (-9 + 4) c_6 \cos 3t = 25 \cos 3t \rightarrow c_6 = \frac{25}{-5} = -5$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos t + c_2 \sin t - 25/8 \cos 3t \\ c_3 \cos 2t + c_4 \sin 2t - 5 \cos 3t \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \right\} \text{general soln}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -c_1 \sin t + c_2 \cos t + 3(25/8) \sin 3t \\ 2c_3 \sin 2t + 2c_4 \cos 2t + 3(5) \sin 3t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 - 25/8 \\ c_3 - 5 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 - 25/8 \\ c_3 - 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \quad \begin{matrix} c_1 = \frac{1}{2} + \frac{25}{8} = \frac{29}{8} \\ c_3 = -\frac{1}{2} + 5 = \frac{9}{2} \end{matrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix} \rightarrow \begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \rightarrow \begin{matrix} c_2 = 1/2 \\ c_4 = 1/4 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 29/8 \cos t + 1/2 \sin t - 25/8 \cos 3t \\ 9/2 \cos 2t + 1/4 \sin 2t - 5 \cos 3t \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -25/8 \\ -5 \end{bmatrix} = \begin{bmatrix} -25/8 + 5 \\ -25/8 - 5 \end{bmatrix} = \begin{bmatrix} 15/8 \\ -65/8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{29}{8} \cos t + \frac{1}{2} \sin t - \frac{25}{8} \cos 3t \\ \frac{9}{2} \cos 2t + \frac{1}{4} \sin 2t - 65/8 \cos 3t \end{bmatrix} \quad \leftarrow \text{scalar form of soln}$$

$$= \underbrace{\left(\frac{29}{8} \cos t + \frac{1}{2} \sin t \right)}_{A_1 \cos(t - \delta_1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\left(\frac{9}{2} \cos 2t + \frac{1}{4} \sin 2t \right)}_{A_2 \cos(t - \delta_2)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos 3t \begin{bmatrix} 15/8 \\ -65/8 \end{bmatrix}$$

$$A_1 = \frac{1}{8} \sqrt{29^2 + 4^2} \approx 3.66$$

$$\delta_1 = \arctan \frac{4}{29} \approx 3.9^\circ$$

$$A_2 = \frac{1}{4} \sqrt{18^2 + 1} \approx 4.51$$

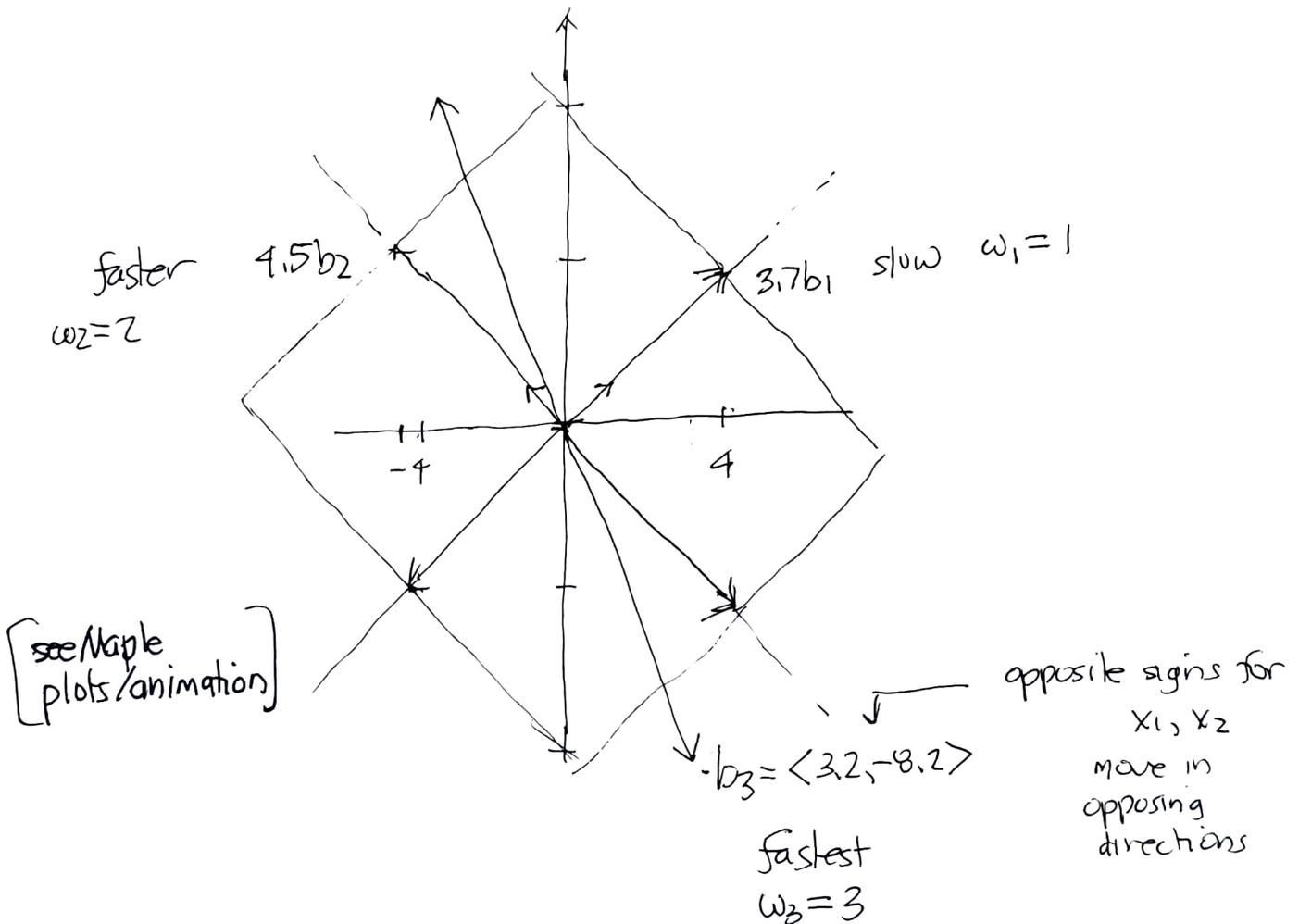
$$\delta_2 = \arctan \left(\frac{1}{18} \right) \approx 0.009^\circ$$

$$\frac{5}{8} \begin{bmatrix} 3 \\ -13 \end{bmatrix} \approx \begin{bmatrix} 3.125 \\ -8.125 \end{bmatrix}$$

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Result: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{A_1 \cos(t - \delta_1) b_1 + A_2 \cos(2t - \delta_2) b_2}_{\text{homogeneous soln which "bridges" initial conditions to those of the "steady state soln" (would exponentially decay if any damping were present so "transient")}} + \underbrace{\cos 3t b_3}_{\text{"steady state" response}}$



soln is a superposition of 3 independent oscillations along b_1, b_2, b_3 at different frequencies.

Because 1, 2, 3 have rational ratios, they share a common period in this case $T = 2\pi$.

Different initial conditions only change the homogeneous soln, not the response function.

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To explore resonance with this system we just replace $50 \cos 3t$ by $50 \cos \omega t$ ($\omega \geq 0$) and repeat, ignoring the homogeneous solution to find the response function.

The decoupled equations are now:

$$y_1'' + y_1 = 25 \cos \omega t \quad y_{1p} = c_5 \cos \omega t \xrightarrow{\text{backsub}} c_5(-\omega^2 + 1) = 25$$

$$y_2'' + 4y_2 = 25 \cos \omega t \quad y_{2p} = c_6 \cos \omega t \xrightarrow{\text{backsub}} c_6(-\omega^2 + 4) = 25$$

$$c_5 = \frac{25}{1-\omega^2} \quad c_6 = \frac{25}{4-\omega^2}$$

$$\begin{bmatrix} x_{1p} \\ x_{2p} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_{1p} \\ y_{2p} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 25/(1-\omega^2) \\ 25/(4-\omega^2) \end{bmatrix} \cos \omega t$$

$$\begin{bmatrix} 25 \left(\frac{1}{1-\omega^2} - \frac{1}{4-\omega^2} \right) \\ 25 \left(\frac{1}{1-\omega^2} + \frac{1}{4-\omega^2} \right) \end{bmatrix} = \begin{bmatrix} \frac{25 \cdot 3}{(1-\omega^2)(4-\omega^2)} \\ \frac{25(5-2\omega^2)}{(1-\omega^2)(4-\omega^2)} \end{bmatrix} = \frac{25}{(1-\omega^2)(4-\omega^2)} \begin{bmatrix} 3 \\ 5-2\omega^2 \end{bmatrix}$$

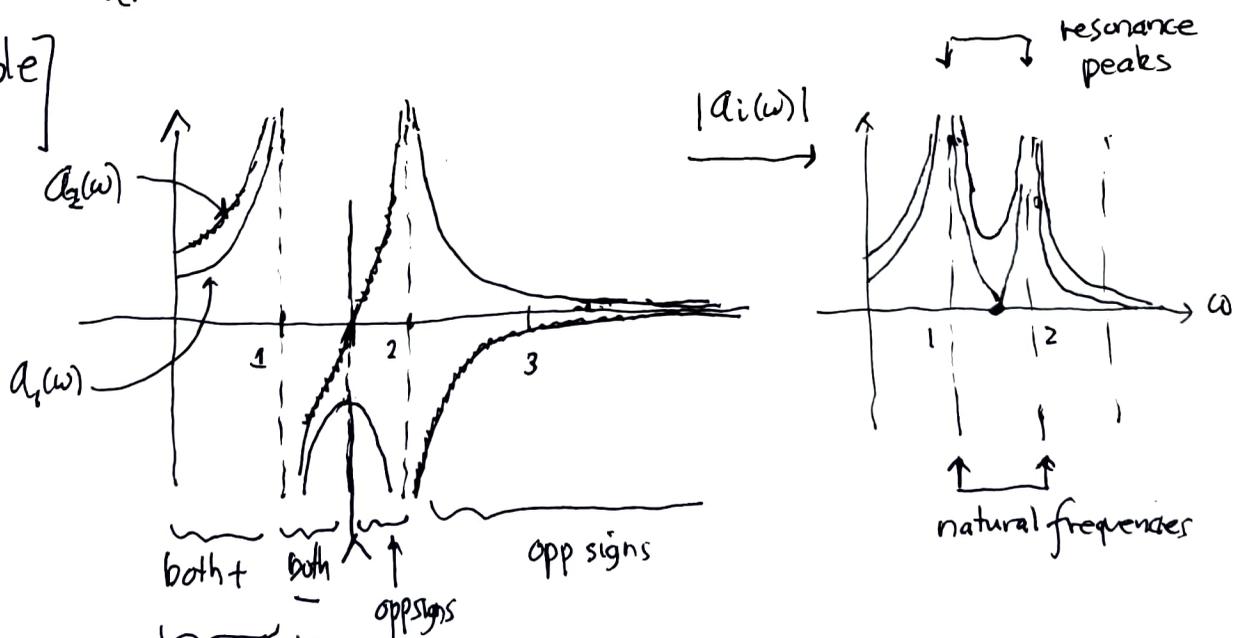
$$\equiv \begin{bmatrix} a_1(\omega) \\ a_2(\omega) \end{bmatrix}$$

$$= \begin{bmatrix} a_1(\omega) \\ a_2(\omega) \end{bmatrix} \cos \omega t$$

"signed amplitudes"

$a_1(\omega) > 0$ but $a_2(\omega) = 0$ when $\omega^2 = 5/2$, $\omega = \sqrt{5/2} \approx 1.58$

[see Maple plots!]



can keep up with driving force

flips through 90° to 180° passing thru resonance

(etc. changes direction sign passing thru each equilibrium)