

7.5a Extending the eigenvector decoupling technique

(1)

The same linear change of variables can be extended to the nonhomogeneous case and to second order DEs.

1st order: $\underline{x}' = \underbrace{A}_{n \times n} \underline{x} \longrightarrow \underline{x}' = Ax + F$
 nonhomogeneous case: driving vector function F

2nd order: $\underline{x}'' = A \underline{x} \longrightarrow \underline{x}'' = Ax + F$

change of variables:
 $\underline{x} = B\underline{y}, \underline{y} = B^{-1}\underline{x} \rightarrow A_B = B^{-1}AB = \text{diagonal}$ is eigenbasis

example:

$$\underline{x}'' = Ax \rightarrow (B\underline{y})'' = AB\underline{y} + F$$

$$B^{-1}(B\underline{y}'') = AB\underline{y} + F]$$

$$\underline{y}'' = \underbrace{B^{-1}AB}_{AB} \underline{y} + \underbrace{B^{-1}F}_{F_B}$$

new components of vector F
 analogous to
 $\underline{y} = B^{-1}\underline{x}$ new components
 of position vector \underline{x}
 $B^{-1}F \equiv F_B$

$$\underline{y}'' = A_B \underline{y} + F_B$$

$$\underline{y}_i'' = \lambda_i \underline{y}_i + (F_B)_i$$

decoupled DEs, solve by Chap 5 methods:
 nonhom case: method of undetermined coeffs

$$\underline{y} = \underline{y}_h + \underline{y}_p : \quad \underline{y}_{hi} = c_i e^{\lambda_i t}$$

interesting case:
 oscillations

$$\lambda_i = -\omega_i^2 < 0 \rightarrow \underline{y}_{fi} = c_{1i} \cos \omega_i t + c_{2i} \sin \omega_i t$$

$F_i = F_{i0} \cos \omega t$, resonance exploration.

$$\underline{x} = B(\underline{y}_h + \underline{y}_p) = \underbrace{B\underline{y}_h}_{\underline{x}_h} + \underbrace{B\underline{y}_p}_{\underline{x}_p}$$

natural oscillations
 of coupled systems
 (undamped!)

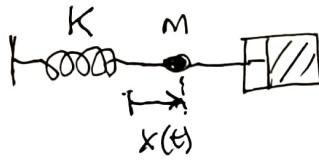
response vector solns
 when driven by oscillating forcing functions

7.5a

Coupled mass spring systems

①

A single mass spring system has a simple description

 $n=1$ mass-spring $x(t)$ = scalar displacement from equilibrium at $x=0$

$$mx'' + cx' + kx = F$$

mass 1 damping Hooke's law driving force
acceleration

↓ NO DAMPING

$$mx'' + kx = F \rightarrow 0 \text{ no driving force}$$

$$x'' + \frac{k}{m}x = 0 \quad \text{oscillations at natural frequency } \omega_0$$

 $n > 1$ coupled mass spring systems

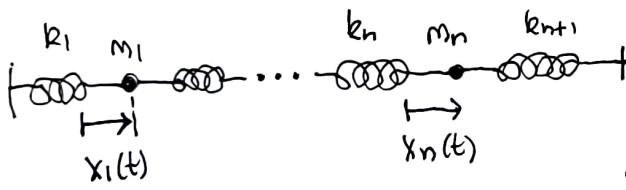
$$x = \langle x_1, \dots, x_n \rangle$$

1-d motion:

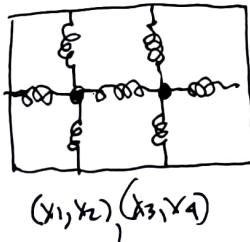
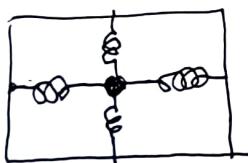
$$Mx'' + Cx' + Kx = F$$

constant $n \times n$
coefficient
matrices

matrix equation of motion


 $x = \langle x_1, \dots, x_n \rangle$ vector variable of displacements of n masses from equilibrium

2-d motion:



etc.

We will only consider 2 or 3 mass spring systems in 1-d motion with only 2×2 and 3×3 matrix DE systems.

First we study the undriven motion (homogeneous DE system) to discover the "natural oscillation modes" at the natural frequencies of the coupled systems, [with no damping present.]

Then we study the response of the system to being driven by a sinusoidal forcing function at a variable frequency, starting with a concrete value to get the hang of things.

Our eigenvector decoupling approach is all we need for this.

7.5a Coupled mass spring systems

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undamped case: $M \ddot{X}'' = -K \dot{X} + F$

$$\text{explicitly: } \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & m_n \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_n \end{bmatrix} = \begin{bmatrix} m_1 \ddot{x}_1'' \\ \vdots \\ m_n \ddot{x}_n'' \end{bmatrix} = -K \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$$

diagonal mass matrix "restoring forces": Hooke's law matrix applied forces

Divide rows by masses to isolate second derivatives:

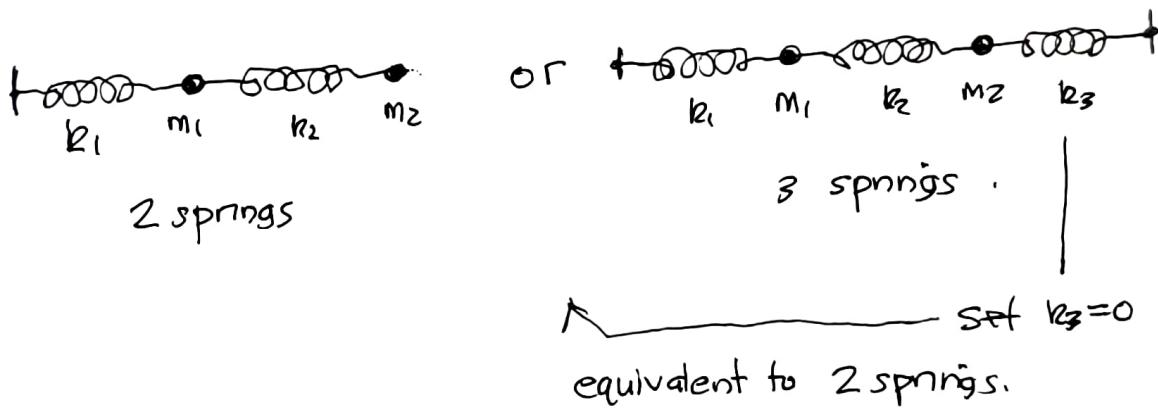
$$\begin{bmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_n \end{bmatrix}'' = \underbrace{(-M^{-1}K)}_A \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} F_1/m_1 \\ \vdots \\ F_n/m_n \end{bmatrix} \quad \Leftrightarrow \boxed{\ddot{X}'' = AX + f}$$

accelerations f

$f = 0$: "natural behavior" of undriven system

$f \neq 0$: drive system with applied forces (even constant forces useful)

simplest case: linear motion of two coupled mass-spring systems with 2 or 3 springs

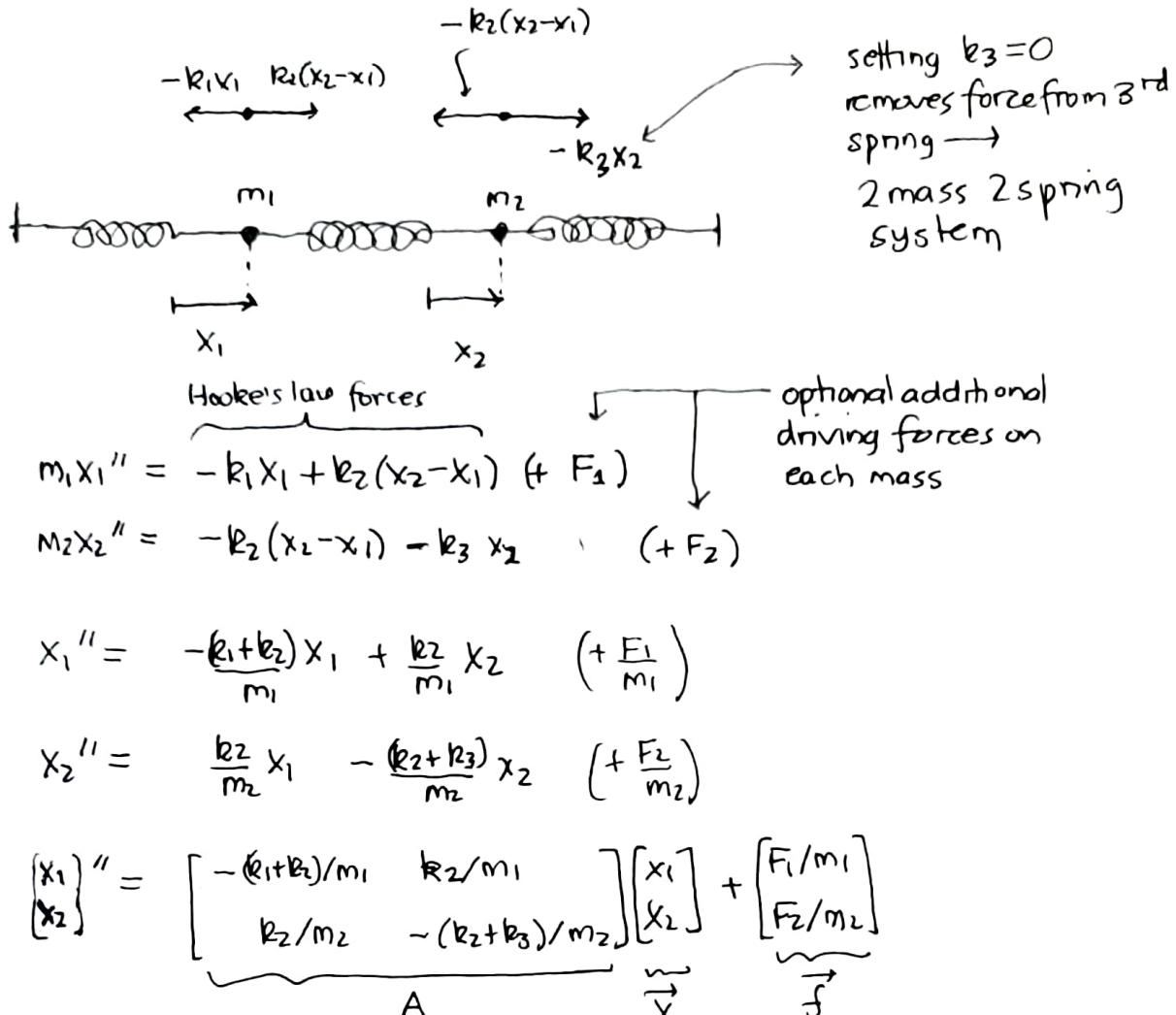
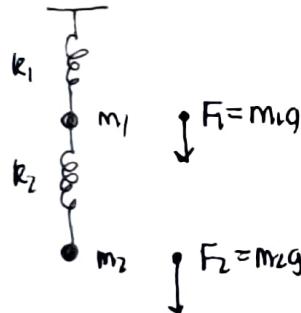


x_1, x_2 are displacements from equilibrium positions

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2 mass 3 spring system: equations of motion

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2 mass 2 spring system with gravity (set k3=0)

Remove spring 3, hung from ceiling, constant gravitational force pulls down, look for equilibrium solns $\vec{x} = \vec{x}_0$ (constant)

$$\vec{x}'' = A\vec{x} + \vec{f}$$

$$0 = \vec{x}_0'' = A\vec{x}_0 + \vec{f} \rightarrow \text{solve } \vec{x}_0 = -A^{-1}\vec{f} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

equilibrium positions.

$$\vec{f} = \langle g, g \rangle$$

$$\text{Note } (\vec{x} - \vec{x}_0)'' = \vec{x}'' = A\vec{x} + \vec{f} = A(\vec{x} - \vec{x}_0 + \vec{x}_0) + \vec{f}$$

$$= A(\vec{x} - \vec{x}_0) + \underbrace{A\vec{x}_0 + \vec{f}}_0 = 0 \text{ for equilibrium}$$

$$\text{so } (\vec{x} - \vec{x}_0)'' = A(\vec{x} - \vec{x}_0)$$

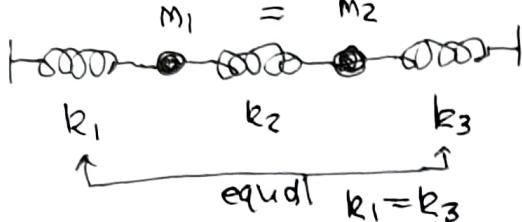
Departures from equilibrium satisfy the homogeneous equations which are oscillations.

7.5a coupled mass spring systems

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2 mass 3 spring system example:

$$m_1 = m_2 \quad (\text{symmetric about center})$$



CHOOSE:

$$\frac{k_1}{m_1} = 1, \underbrace{\frac{k_2}{m_2} = \frac{3}{2}}_{\text{stronger spring in center}}, \underbrace{\frac{k_3}{m_3} = 1}_{\text{stronger spring in center}}$$

(leads to integer eigenvalues of A!)

$$A = \begin{bmatrix} -(\frac{k_1+k_2}{m_1}) & \frac{k_2}{m_1} \\ \frac{k_2}{m_1} & -(\frac{k_2+k_3}{m_1}) \end{bmatrix}$$

$$= \begin{bmatrix} -(1+\frac{3}{2}) & \frac{3}{2} \\ \frac{3}{2} & -(1+\frac{3}{2}) \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 3/2 & -5/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solve by eigenvector approach

$$x_1'' = (-5x_1 + 3x_2)/2$$

$$\text{add initial conditions: } \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x_2'' = (3x_1 - 5x_2)/2$$

$$\lambda = -1, -4$$

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \langle b_1 | b_2 \rangle$$

$$B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1} A B = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$x = By = y_1 b_1 + y_2 b_2, \quad y = B^{-1} x$$

$$x'' = Ax \rightarrow B^{-1}(By)'' = B^{-1}A(By) \rightarrow y'' = A_B y$$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_1 \\ -4y_2 \end{bmatrix} \rightarrow y_1'' = -y_1 \rightarrow y_1'' + y_1 = 0 \rightarrow y_1 = c_1 \cos t + c_2 \sin t$$

$$y_2'' = -4y_2 \rightarrow y_2'' + 4y_2 = 0 \rightarrow y_2 = c_3 \cos 2t + c_4 \sin 2t$$

$$\text{natural frequencies: } \omega_1 = \sqrt{-\lambda_1} = 1, \quad \omega_2 = \sqrt{-\lambda_2} = 2$$

$$\text{periods: } T_1 = 2\pi, \quad T_2 = 2\pi/2 = \pi$$

7.5a

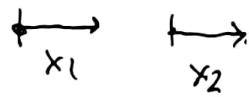
coupled mass spring systems

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general soln:

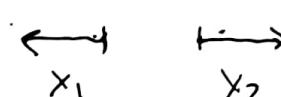
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ c_3 \cos 2t + c_4 \sin 2t \end{bmatrix} = \underbrace{(c_1 \cos t + c_2 \sin t)}_{A_1 \cos(t-\delta_1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{(c_3 \cos 2t + c_4 \sin 2t)}_{A_2 \cos(2t-\delta_2)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

slow mode $\omega_1 = 1$ fast mode $\omega_2 = 2$
 equal amplitudes, same direction equal amplitudes opposite directions



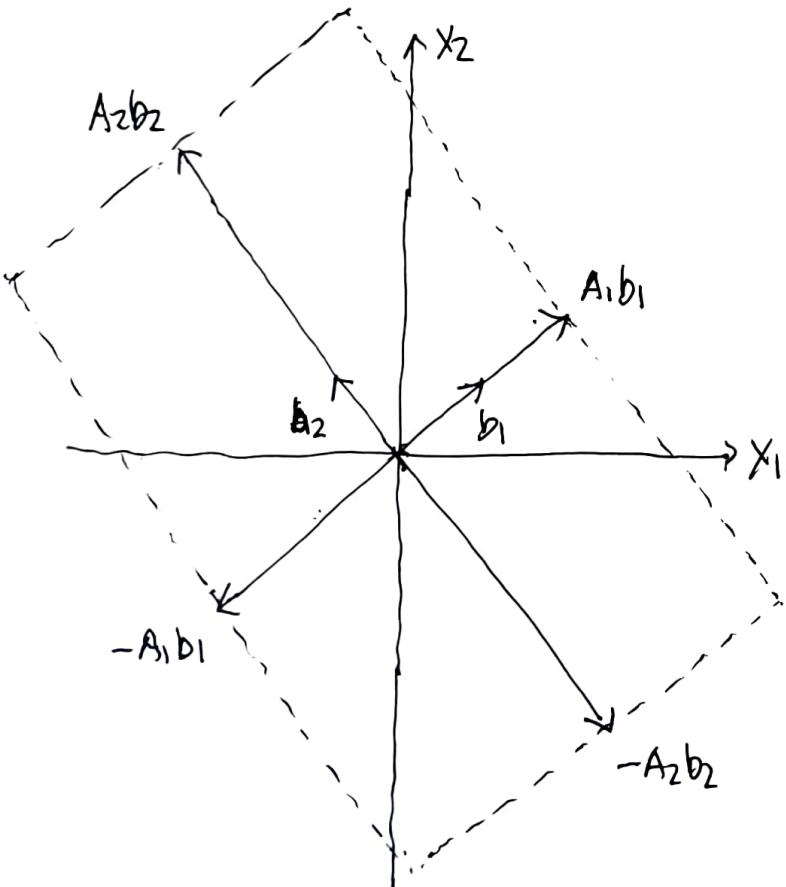
$$\frac{x_2}{x_1} = 1$$

"tandem mode"



$$\frac{x_2}{x_1} = -1$$

"accordian mode"



$$x = \underbrace{\cos(\omega_1 t - \delta_1)}_{\pm 1} [A_1 b_1] + \underbrace{\cos(\omega_2 t - \delta_2)}_{\pm 1} [A_2 b_2]$$

bits edge of rectangle
which confines motion
inside

7.5a coupled mass spring systems

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IVP solution:

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = B \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ c_3 \cos 2t + c_4 \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ y_2' \end{bmatrix} = B \begin{bmatrix} -c_1 \sin t + c_2 \cos t \\ -2c_3 \sin 2t + 2c_4 \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ y_2(0) \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

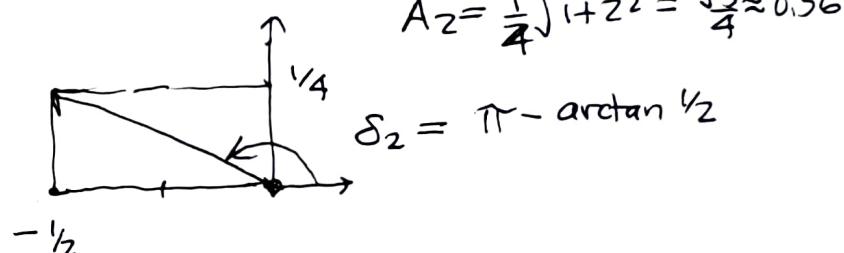
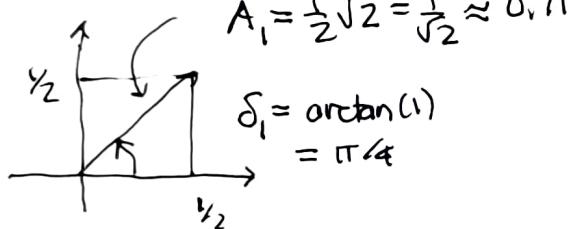
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ y_2'(0) \end{bmatrix} = B \begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix}$$

$$\begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix} = B^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \quad \text{extra step.}$$

Backsub into Y:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2} \sin t \\ -\frac{1}{2} \cos 2t + \frac{1}{4} \sin 2t \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \cos(t - \pi/4) \\ \sqrt{5}/4 \cos(2t - \arctan 1/2) \end{bmatrix} = \begin{bmatrix} A_1 \cos(t - \delta_1) \\ A_2 \cos(2t - \delta_2) \end{bmatrix}$$

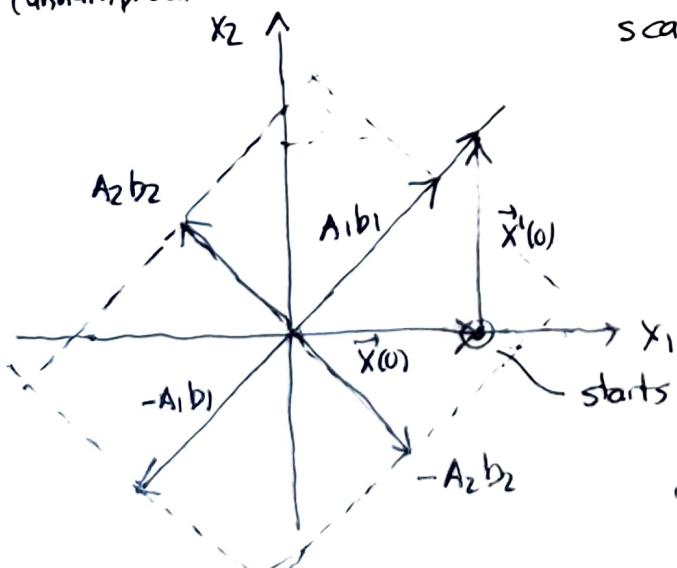


$$\vec{x} = \underbrace{\frac{1}{\sqrt{2}} \cos(t - \delta_1) \vec{b}_1}_{\text{slow mode}} + \underbrace{\frac{\sqrt{5}}{4} \cos(2t - \delta_2) \vec{b}_2}_{\text{fast mode}}$$

vector form of
soln showing
2 modes

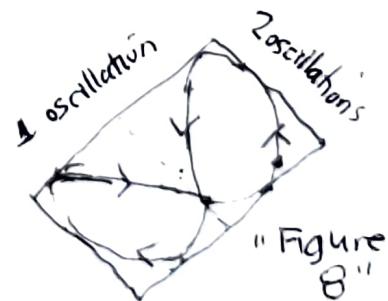
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2} \sin t \\ -\frac{1}{2} \cos 2t + \frac{1}{4} \sin 2t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2} \sin t + \frac{1}{2} \cos 2t - \frac{1}{4} \sin 2t \\ \frac{1}{2} \cos t + \frac{1}{2} \sin t - \frac{1}{2} \cos 2t + \frac{1}{4} \sin 2t \end{bmatrix}}_{\text{scalar form showing } x_1, x_2}$$

(unmultiplied) matrix form



starts out here, moving up

stays inside rectangle
oscillates faster in b2 direction

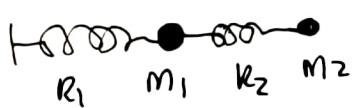


"Figure 8"

7.5a)

Coupled mass spring systems

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Now you repeat for (set $k_3 = 0$):

$$m_1 = 2 \quad k_1 = 4 \\ m_2 = 1 \quad k_2 = 2$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -k_2/m_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\begin{cases} \lambda = -1, -4 \\ B = \begin{bmatrix} y_2 & -1 \\ 1 & 1 \end{bmatrix} = \langle b_1 | b_2 \rangle, \quad A_B = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \end{cases}$$

- Express the decoupled DEs for y_1, y_2 in matrix and then scalar form.
- Solve them for general soln for y_1, y_2 .
- Impose initial conditions on $x = By$ solving both using B^{-1} to determine values of arbitrary constants.
- Backsub into x to obtain TVP soln. (check with Maple!)
- Express y_1, y_2 in phase-shifted cosine form, identifying the amplitudes A_i and phase-shifts δ_i .
- Make a diagram of $b_1, b_2, \pm A_i b_i$ and the parallelogram with the latter axes; namely with corners $\pm A_1 b_1 \pm A_2 b_2$. Draw in $x(0)$ and $x'(0)$ (initial pt at tip of $x(0)$).
- Plot X_2 versus X_1 .
- Plot $x_1(t), x_2(t)$ versus t .

> plot([x1(t), x2(t), t=0..2π], color=[red, blue]) x_2 versus x_1

> plot([x1(t), x2(t)], t=0..2π, color=[red, blue]) x_1, x_2 versus t

↑
common period of two oscillations