

7.5c Damped harmonic oscillator systems

(1)

$$\begin{aligned} m_1 x_1'' + \gamma_1 x_1' &= a_{11} x_1 + a_{12} x_2 + F_1 \\ m_2 x_2'' + \gamma_2 x_2' &= a_{21} x_1 + a_{22} x_2 + F_2 \end{aligned}$$

damping term Hooke's law forces applied forces

Put in standard form: divide by masses, define $k_{10} = \gamma_1/m_1$, $k_{20} = \gamma_2/m_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' + \begin{bmatrix} k_{10} x_1' \\ k_{20} x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} k_{10} & 0 \\ 0 & k_{20} \end{bmatrix}}_{\text{diagonal}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'$$

next transform to new basis of eigenvectors:

$$x = By, y = B^{-1}x$$

$$\begin{bmatrix} x'' + \begin{bmatrix} k_{10} & 0 \\ 0 & k_{20} \end{bmatrix} x' = Ax + F \end{bmatrix}$$

$$B^{-1} \left[(By)'' + \begin{bmatrix} k_{10} & 0 \\ 0 & k_{20} \end{bmatrix} (By)' \right] = A(By) + F$$

$$y'' + \underbrace{B^{-1} \begin{bmatrix} k_{10} & 0 \\ 0 & k_{20} \end{bmatrix} B}_{\text{diag}(x_1, x_2)} y' = \underbrace{(B^{-1}AB)}_{A_B} y + \underbrace{B^{-1}F}_{F_B}$$

not diagonal unless $k_{10} = k_{20} = k_0$, then

$$B^{-1}(kI)B = k \underbrace{B^{-1}IB}_{I} = kI \quad \text{still diagonal}$$

$$y'' + k_0 I y' = A_B y + F_B \quad \text{decouple:}$$

$$y_i'' + k_0 y_i' - \underbrace{\gamma_i y_i}_{+ \omega_0^2} = (F_B)_i \quad y_i = \text{decaying exponential amplitude oscillation (Chapter 5)}$$

This allows us to investigate the effects of damping with the same eigenvectors but new eigenfrequencies and exponential decay rates.

But otherwise?

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We need the "reduction of order" method which introduces new variable names for derivatives so that all the equations can be written as first order DEs.

variables: x_1, x_2

introduce the velocity variables $v_1 = x_1'$, $v_2 = x_2'$ (2 extra cons!) as new unknowns, so $v_1' = x_1''$, $v_2' = x_2''$
converts second derivatives to first derivatives.

$$x_1'' + \gamma_1 x_1' = a_{11}x_1 + a_{12}x_2 + F_1$$

$$x_2'' + \gamma_2 x_2' = a_{21}x_1 + a_{22}x_2 + F_2$$

$$\downarrow$$

$$\downarrow$$

$$v_1' + \gamma_1 v_1 = a_{11}x_1 + a_{12}x_2 + F_1$$

$$v_2' + \gamma_2 v_2 = a_{21}x_1 + a_{22}x_2 + F_2$$

→ $x_1' = v_1$
 $x_2' = v_2$
 $v_1' = -\gamma_1 v_1 + a_{11}x_1 + a_{12}x_2 + F_1$
 $v_2' = -\gamma_2 v_2 + a_{21}x_1 + a_{22}x_2 + F_2$

} 4 first order eqns in
4 variables x_1, x_2, v_1, v_2
rename x_3, x_4

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} & x_3 & & x_4 \\ & & & \\ q_{11}x_1 + q_{12}x_2 & -\gamma_1 x_3 & & \\ q_{21}x_1 + q_{22}x_2 & & -\gamma_2 x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ q_{11} & q_{12} & -\gamma_1 & 0 \\ q_{21} & q_{22} & 0 & -\gamma_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \end{bmatrix}$$

$$x' = Ax + f$$

→ find eigenvectors, eigenvalues of 4×4 matrix, decouple system, solve

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When $\gamma_1 = \gamma_2 = 0$, damping absent:

$$A = \begin{bmatrix} \emptyset_2 & I_2 \\ A & \emptyset_2 \end{bmatrix}$$

$$\emptyset_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find eigenvalues:

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = \pm \sqrt{\lambda_1}, \pm \sqrt{\lambda_2} = \pm i\omega_1, \pm i\omega_2$$

and the solution

$$\begin{aligned} \chi = \begin{bmatrix} X \\ X' \end{bmatrix} &= e^{i\omega_1 t} b_1 + e^{-i\omega_1 t} b_2 + e^{i\omega_2 t} b_3 + e^{-i\omega_2 t} b_4 \\ &= \dots \\ &= \left[\begin{array}{l} (c_1 \cos \omega_1 t + c_2 \sin \omega_1 t) b_1 \\ (-\omega_1 c_1 \sin \omega_1 t + \omega_1 c_2 \cos \omega_1 t) b_1 \end{array} \right] + \left[\begin{array}{l} (c_3 \cos \omega_2 t + c_4 \sin \omega_2 t) b_2 \\ (-\omega_2 c_3 \sin \omega_2 t + \omega_2 c_4 \cos \omega_2 t) b_2 \end{array} \right] \end{aligned}$$

last 2 components are just the derivative of the first two!

we recover soln we already knew how to derive.

when damping is present in general:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} A_1 e^{-K_1 t} \cos(\omega_1 t - \delta_1) \\ A_2 e^{-K_2 t} \cos(\omega_2 t - \delta_2) \end{bmatrix} + \begin{bmatrix} A_3 e^{-K_1 t} \cos(\omega_1 t - \delta_3) \\ A_4 e^{-K_2 t} \cos(\omega_2 t - \delta_4) \end{bmatrix}$$

↑ relative phaseshifts

two decay rates $K_1, K_2 \rightarrow \sigma_1, \sigma_2$ two decay times

two frequencies ω_1, ω_2

both transient solutions.

can then consider resonance exploration driving this system at frequency ω .

If weak damping get big response near natural frequencies of system.

end of story!