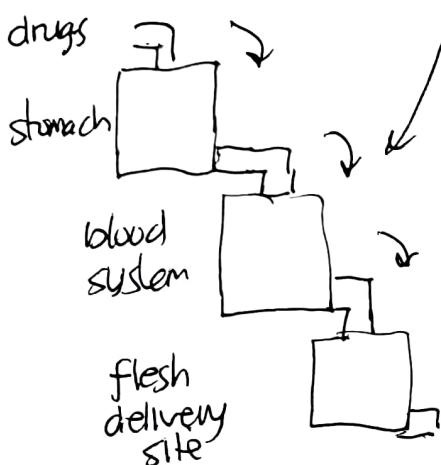
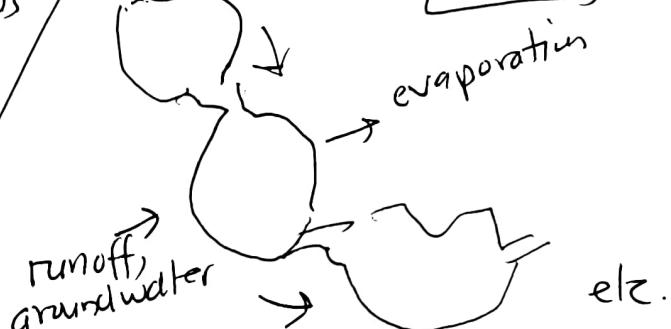


7.3d

Multiple mixing tanks = compartmental analysis

①

In the same way mass-spring systems are merely a concrete representation of a wide variety of systems which undergo damped oscillations, mixing tank problems are representative of a broad spectrum of systems involving coupled exponential behavior. Not just those ugly tanks and pipe complexes we can see from I-95 in NJ on the way to NYC: pharmacokinetics in drug delivery modeling, environmental & epidemiology modeling etc.

Examples:Big Pharmaor continuous  
intravenous  
feedmultiple Great Lakes  
and pollution  
modelingepidemiology + epidemic/pandemic modeling

"compartments": population groups

disease free, contact with disease,  
active victims, etc.

We do toy problems of a solution of a "solute" in a "solvent" with a certain concentration, like "brine" = saltwater  $\Rightarrow$  salt = solute  $\Rightarrow$  solution = salt water  
water = solvent

We have some intuition about concentrations of solutes in a solution, so these make good test problems to understand this class of systems.

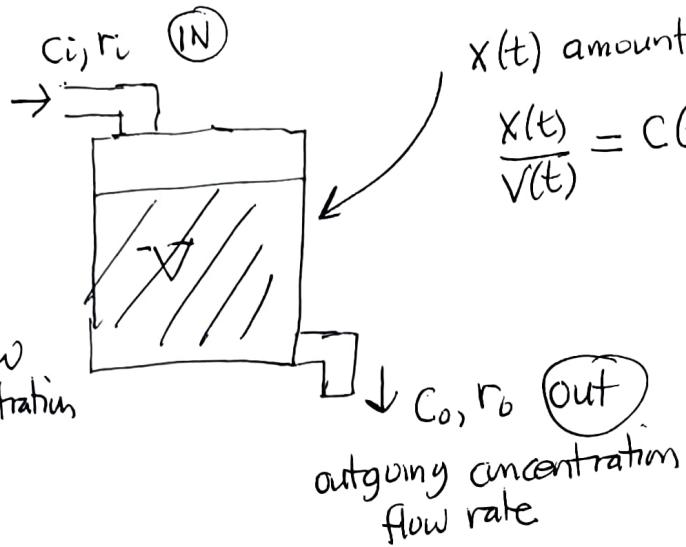
7.3d

Multiple mixing tanks

(2)

Single tank

constant incoming flow rate, concentration



$$C_o(t) = \frac{X(t)}{V(t)} \quad \text{assumption of thorough mixing on much shorter time scale}$$

product  $C_i r_i = \frac{\text{time}}{\text{volume}} \text{rate of change } X(t) \text{ for incoming flow}$

$$\frac{\text{solute}}{\text{volume}} \frac{\text{volume}}{\text{time}}$$

product  $C_o r_o = \text{time rate of change } X(t) \text{ for outgoing flow}$

$$= \left( \frac{X(t)}{V(t)} \right) \cdot r_o = \frac{r_o}{V(t)} X(t) = \frac{r_o}{V_0} X(t) \equiv k X(t)$$

$$\frac{dx}{dt} = \underbrace{r_i c_i}_{\text{constant}} - \underbrace{r_o c_o}_{k X} \quad (\text{rate in minus rate out})$$

Now connect up tanks in series.

Two tanks...

Three tanks...

We will examine the latter case.

What goes out of one goes into the next.

For "flow thru"

$$r_i = r_o = r$$

$$V(t) = \text{constant} = V_0$$

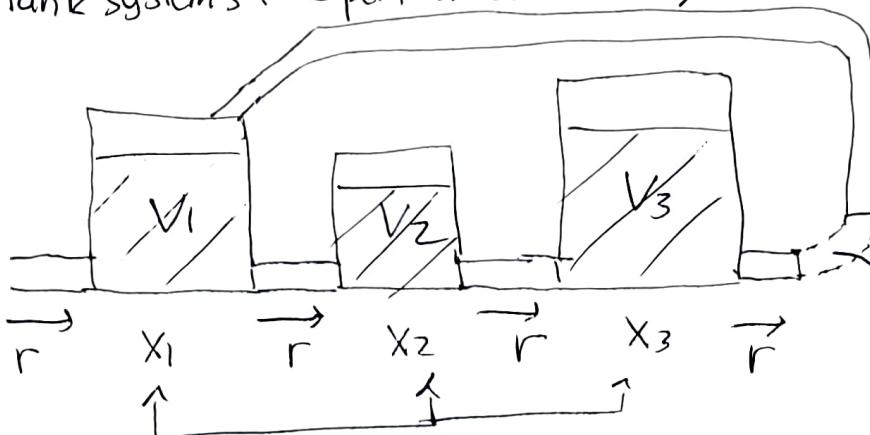
7.3d

Multiple mixing tanks

(3)

Three tank systems: open and closed, flow thru

(IN)

 $c_i = 0$ ,  
only  
solvent


$$R_i = \frac{r}{V_i}$$

no input      closed:

$$\begin{aligned}\frac{dx_1}{dt} &= g R_i - R_1 x_1 \quad (+ R_3 x_3) \\ \frac{dx_2}{dt} &= R_1 x_1 - R_2 x_2 \\ \frac{dx_3}{dt} &= R_2 x_2 - R_3 x_3\end{aligned}$$

open case:  
If  $C_i \neq 0$   
we get nonhom.  
system with  
constant  
driving  
vector  
function

What will happen? we can guess.

These problems start with  $x_1(0) = x_0$ ,  $x_2(0) = 0$ ,  $x_3(0) = 0$ :  
Only solvent in the second two tanks and only solvent incoming.

The solvent flows in and in the closed system equalizes the solute concentrations in the 3 tanks so the proportion of solute is proportional to the volume eventually..

In the open system solute is transferred from the first to the remaining tanks where the amount initially increases but eventually the solvent flushes them all out and the solutes all go to zero.

In the open system if we add incoming solute at a constant rate, eventually all the tanks will equilibrate to that concentration.

Equal concentration means amount of solute is proportional to volume.

7.3d Multiple mixing tanks

(4)

Explicit example with complex eigenvalues

$$r = 10, v_1 = 50, v_2 = 25, v_3 = 50 \rightarrow [k_1, k_2, k_3] = \left[ \frac{10}{50}, \frac{10}{25}, \frac{10}{50} \right] \\ = \left[ \frac{1}{5}, \frac{2}{5}, \frac{1}{5} \right]$$

closed system:  $x_0 = 100$  (total solvent)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -k_1 & 0 & k_3 \\ k_1 & -k_2 & 0 \\ 0 & k_2 - k_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -\frac{(1+i)}{2} & \frac{1-i}{2} \\ \frac{1}{2} & -\frac{(1-i)}{2} & -\frac{(1+i)}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\lambda = 0, -\frac{2+i}{5}, -\frac{2-i}{5}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 b_1 + c_2 e^{\frac{-2+i}{5}t} b_2 + c.c.$$

$$e^{-2t} \left( \cos \frac{t}{5} + i \sin \frac{t}{5} \right) \begin{bmatrix} -(1+i)/2 \\ -(1-i)/2 \\ 1 \end{bmatrix} = e^{-2t} \begin{bmatrix} -\frac{1}{2}(c+s) - \frac{i}{2}(c-s) \\ -\frac{1}{2}(c+s) + \frac{i}{2}(c-s) \\ c + i s \end{bmatrix}$$

$$= e^{-2t} \underbrace{\begin{bmatrix} -\frac{1}{2}(c-s) \\ -\frac{1}{2}(c+s) \\ c \end{bmatrix}}_{X_1} + i e^{-2t} \underbrace{\begin{bmatrix} -\frac{1}{2}(c+s) \\ +\frac{1}{2}(c-s) \\ s \end{bmatrix}}_{X_2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 b_1 + c_2 X_1 + c_3 X_2$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1/2 \\ +1/2 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1 & 1 & 0 \end{bmatrix}}_B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 2 & 2 \\ -2 & -2 & 3 \\ -4 & 6 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \frac{100}{5} \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 40 \\ -40 \\ -80 \end{bmatrix}$$

7.3d

Multiplemixing tanks

(5)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 40 \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} - 40e^{-\frac{2t}{5}} \begin{bmatrix} -\frac{1}{2}\cos\frac{t}{5} + \frac{1}{2}\sin\frac{t}{5} \\ -\frac{1}{2}\cos\frac{t}{5} - \frac{1}{2}\sin\frac{t}{5} \\ \cos\frac{t}{5} \end{bmatrix} - 80e^{-\frac{2t}{5}} \begin{bmatrix} -\frac{1}{2}\cos\frac{t}{5} - \frac{1}{2}\sin\frac{t}{5} \\ \frac{1}{2}\cos\frac{t}{5} - \frac{1}{2}\sin\frac{t}{5} \\ \sin\frac{t}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ 20 \\ 40 \end{bmatrix} + e^{-\frac{2t}{5}} \begin{bmatrix} (20+40)\cos\frac{t}{5} + (-20+40)\sin\frac{t}{5} \\ (20-40)\cos\frac{t}{5} + (20+40)\sin\frac{t}{5} \\ -40\cos\frac{t}{5} - 80\sin\frac{t}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ 20 \\ 40 \end{bmatrix} + e^{-\frac{2t}{5}} \begin{bmatrix} 120\cos\frac{t}{5} - 60\sin\frac{t}{5} \\ -20\cos\frac{t}{5} + 80\sin\frac{t}{5} \\ -40\cos\frac{t}{5} - 80\sin\frac{t}{5} \end{bmatrix}$$

100 proportionally distributed

transient taking initial conditions to final state

$$\tau = 2.5 \rightarrow 5\tau = 12.5 \text{ decay window duration}$$

[see Maple plot  
then open case]

oscillation period:

$$T = \frac{2\pi}{\gamma} = 10\pi \approx 31.4$$

$\frac{5\tau}{T} \approx 0.40$  so only about  
a bit less than half an oscillation  
is visible in a decay window

Qualitatively real or complex eigenvalues do not look much different  
for the closed case.

(one can show  $\frac{5\tau}{T} \lesssim 0.46$  for all possible parameter values)

7.3d

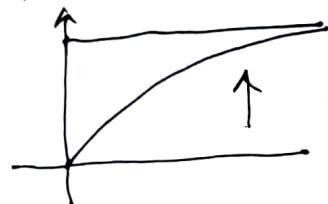
Multiple mixing tanks

(6)

In the open case we always get negative eigenvalues so decreasing exponentials. So how do we get solutions where the amounts in the second two tanks initially grow before falling to zero as the first tank amount flows into them?

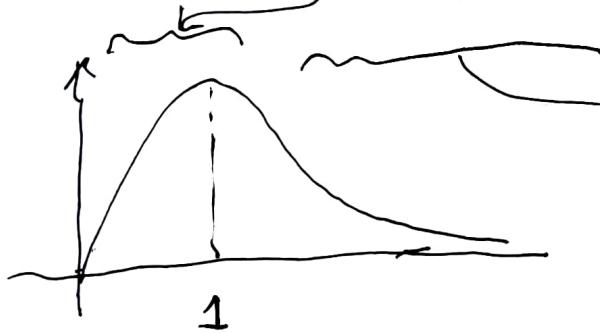
Differences of decaying exponentials can initially grow.

$$x(t) = 1 - e^{-t}, \quad x(0) = 0$$



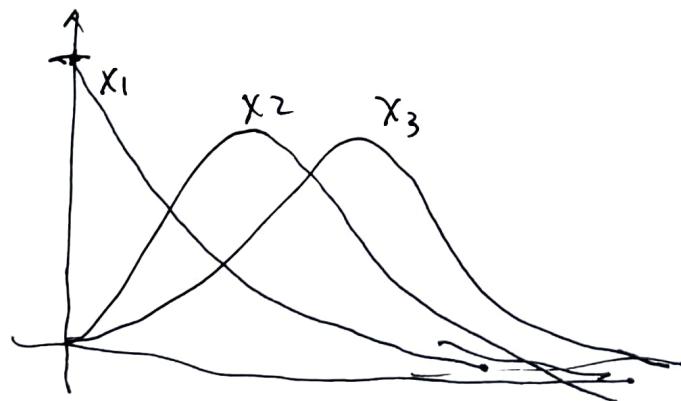
↑ grows from zero because the difference decays

$$x(t) = \underbrace{e^{-t}}_{\text{slower to decrease}} - \underbrace{e^{-2t}}_{\text{gains on it so difference initially decays}}, \quad x(0) = 0$$



finally first exponential begins to decrease appreciably.

3 tank case



eventually all go to zero but initially solute flows from tank 1 to tank 2 and then tank 3 before all decreasing.