7.3c Nonhomogeneous 1st order DE systems
(and eigenvector decoupling extended to higher order)

Our eigenvector decoupling approach also works in the nonhomogeneous case as long as the RHS vector function components are compatible with the method of undetermined coefficients.

We get decoupled nonhomogeneous constant coefficient DEs which we solve independently as in Chapter 5, then return to our original variables. We can also change x' to x'' to solve the corresponding second order DEs.

So vector variable $x = \langle x_1, \ldots, x_n \rangle$

RHS driving vector function: $F = \langle F_1, \ldots, F_n \rangle$, functions only of t

IVP

$$\underbrace{\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n + F_1 \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n + F_n \end{aligned}}_{\text{scalar form}} \qquad x_1(0) = x_{10}, \ldots, x_n(0) = x_{n0}$$

IVP: $\underbrace{x' = Ax + F}_{\text{matrix form}}$, $\quad x(0) = x_0 \qquad$ for vector variable x.

Analyze eigenvalues and eigenvectors of A. If A is diagonalizable over the real or complex numbers, these DEs decouple.

$A \to B = \langle b_1 | \cdots | b_n \rangle \longrightarrow A_B = B^{-1}AB = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$

$\lambda = \lambda_1, \ldots, \lambda_n$

Transform DEs: $\quad x = By, \; y = B^{-1}x \;$ and $\; F = BF_B, \; \underbrace{F_B = B^{-1}F}_{\text{new components of vector}}$

$B^{-1}[(By)' = A(By) + F] \to y' = (B^{-1}AB)y + B^{-1}F$

$y' = A_B y + B^{-1}F$

decoupled DEs: $\boxed{y_i' = \lambda_i y_i + (B^{-1}F)_i}$ for $1 \le i \le n$

solve by 1st order linear algorithm for any driving functions or by method of undetermined coefficients for compatible RHSs.

Then transform back $x = By$.

For the sake of a useful example, consider constant F case.

$y_i' = \lambda_i y_i + f_i \qquad f_i \equiv (B^{-1}F)_i$ constants.

$e^{-\lambda_i t}[\,y_i' - \lambda_i y_i = f_i\,] \longrightarrow \frac{d}{dt}(y_i e^{-\lambda_i t}) = f_i e^{-\lambda_i t}$

$e^{\int -\lambda_i\, dt} = e^{-\lambda_i t}$

$y_i e^{-\lambda_i t} = \int f_i e^{-\lambda_i t}\, dt$ if $\lambda_i \neq 0$!

$= -\frac{f_i}{\lambda_i} e^{-\lambda_i t} + C_i$

[if $\lambda_i = 0$, then $\int f_i\, dt = \underbrace{f_i t}_{y_{ip}} + C_i$]

$y_i = e^{\lambda_i t}\left(-\frac{f_i}{\lambda_i} e^{-\lambda_i t} + C_i\right)$

$= -\frac{f_i}{\lambda_i} + C_i e^{\lambda_i t}$

$= y_{ip} + y_{ih}$

homogeneous solution just acquires additive constant.

$X = By = B(y_h + y_p)$

$= \underbrace{By_h}_{} + \underbrace{By_p}_{}$

$= X_h + X_p$

constant vector, solution of $\underbrace{X_p'}_{=0!} = AX_p + F$

so $AX_p = -F$

$\rightarrow \boxed{X_p = -A^{-1}F}$ (A^{-1} exists if all eigenvalues are nonzero)

just shifts equilibrium soln away from $\vec{0}$.

In fact $(X - X_p)' = X' = AX + \underbrace{F}_{= -AX_p} = A(X - X_p)$ difference satisfies the homogeneous DE.

The textbook directly applies the method of undetermined coefficients to the old vector variable, but our eigenvector decoupling approach also works for higher order DE systems.

Instead we can move on to higher order constant coefficient linear DE systems with only one coefficient matrix, which can be diagonalized to decouple the system exactly as in the first order case.

What is the GENERAL CASE of a constant coefficient linear DE system of second order?

vector variable $X = \langle x_1, \ldots, x_n \rangle$, RHS vector $F = \langle F_1, \ldots, F_n \rangle$ function only of t

$$\boxed{\underbrace{A_2}_{} X'' + \underbrace{A_1}_{} X' + \underbrace{A_0}_{} X = F}$$

three $n\times n$ coeff matrices

<u>complication 1</u> We can only diagonalize one matrix at a time in general except for the very special case that they all share the same eigenbasis (but differing eigenvalues allowed).

<u>complication 2</u> If $\det A_2 = 0$, then A_2^{-1} does not exist so we cannot solve for the highest derivative to put the system into standard form. This leads to "mixed" order DE systems.

suppose $\det A_2 \neq 0$ then

$$A_2^{-1}\left[A_2X'' + A_1X' + A_0X = F\right]$$

$$IX'' + \underbrace{A_2^{-1}A_1}_{B_1} X' + \underbrace{A_2^{-1}A_0}_{B_0} X = \underbrace{A_2^{-1}F}_{f}$$

$$IX'' + BX' + B_0X = f$$

still 2 distinct matrices, no go unless "simultaneously diagonalizable"

unless one term is missing: for example

$$X'' + B_0X = f \rightarrow X'' = -B_0X + f$$

$$\boxed{X'' = AX + f}$$ only difference is an extra prime.

Eigenvector decoupling can handle this case.

7.3c Nonhomogeneous 1st order DE systems, etc.

Both complications can be handled by introducing new variables to denote all derivatives up to but not including the highest derivative of each variable appearing in the system and rewriting the DEs as a 1st order linear DE system in higher dimension with one coefficient matrix which can be diagonalized to decouple the system. This is called $\boxed{\text{"reduction of order"}}$.
We will come back to it after the simpler second order case:

$$x'' = Ax + F$$

$$\Big\downarrow \; x = By,\ y = B^{-1}x$$

$$B^{-1}\left[(By)'' = A(By) + F\right] \to y'' = (B^{-1}AB)y + B^{-1}F$$

$$\boxed{y'' = A_B y + B^{-1}F}$$

decoupled DEs: $y_i'' = \lambda_i y_i + (B^{-1}F)_i$

solve each one independently as in Chapter 5,
then return to $x = By$.

homogeneous case:

$$x'' = Ax \to y'' = A_B y \to y_i'' = \lambda_i y \to \underbrace{y_i'' - \lambda_i y_i = 0}_{\text{standard form}}$$

$y_i = e^{kt}$ trial exponential solns

$y_i'' = k^2 e^{kt}$

$y_i'' - \lambda_i y_i = (k^2 - \lambda_i)e^{kt} = 0 \to k = \pm\sqrt{\lambda_i}$ ← ± square roots of eigenvalues become rate factors now

$e^{kt} = e^{\pm\sqrt{\lambda_i}t}$

gen soln: $y_i = c_{i1} e^{\sqrt{\lambda_i}t} + c_{i2} e^{-\sqrt{\lambda_i}t}$

[complex eigenvalues are out here. negative eigenvalues lead to oscillations]

go back: $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = y_1 b_1 + \cdots + y_n b_n$ 2n modes

initial conditions determine $2n$ constants: c_{i1}, c_{i2}, $1 \le i \le n$

nonhomogeneous case:
just use method of undetermined coefficients on each decoupled DE.

7.3c Nonhomogeneous 1st order linear DEs, etc.

I will work 2 examples both as 1^{st} order and 2^{nd} order systems.

Slowly: $A=\begin{bmatrix}-10&0&0\\10&-5&0\\0&5&-4\end{bmatrix}$ $F=\begin{bmatrix}120\\0\\0\end{bmatrix}$ $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ $\to$ $x'=Ax+F$, $x''=Ax+F$

Quicker: $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $F=\begin{bmatrix}2\\0\end{bmatrix}$ $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $\to$ $x'=Ax+F$, $x''=Ax+F$

$$B=\begin{bmatrix}3/5&0&0\\-6/5&-1/5&0\\1&1&1\end{bmatrix}\qquad B^{-1}=\frac{1}{3}\begin{bmatrix}5&0&0\\-30&-15&0\\25&15&3\end{bmatrix}\qquad B^{-1}F=100\begin{bmatrix}2\\-12\\10\end{bmatrix}$$

$\lambda=-10,-5,-4$ (ordered)

$$A_B=B^{-1}AB=\begin{bmatrix}-10&0&0\\0&-5&0\\0&0&-4\end{bmatrix}$$

$x=By,\ y=B^{-1}x$

$$\begin{bmatrix}y_1'\\y_2'\\y_3'\end{bmatrix}=\begin{bmatrix}-10&0&0\\0&-5&0\\0&0&-4\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}+100\begin{bmatrix}2\\-12\\10\end{bmatrix}=\begin{bmatrix}-10y_1+200\\-5y_2-1200\\-4y_3+1000\end{bmatrix}$$

$y_1'+10y_1=200$ $\quad y_{1h}=c_1e^{-10t}$ $\quad y_{1p}=c_4\to 0+10c_4=200,\ c_4=20$

$y_2'+5y_2=-1200$ $\quad y_{2h}=c_2e^{-5t}$ $\quad y_{2p}=c_5\to 0+5c_5=-1200,\ c_5=-240$

$y_3'+4y_3=1000$ $\quad y_{3h}=c_3e^{-4t}$ $\quad y_{3p}=c_6\to 0+4c_6=1000,\ c_6=250$

constants so trial functions are constants $\qquad$ backsub

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=B\begin{bmatrix}c_1e^{-10t}+20\\c_2e^{-5t}-240\\c_3e^{-4t}+250\end{bmatrix}=\underbrace{B\begin{bmatrix}c_1e^{-10t}\\c_2e^{-5t}\\c_3e^{-4t}\end{bmatrix}}_{x_h}+\underbrace{B\begin{bmatrix}20\\-240\\250\end{bmatrix}}_{x_p=\langle 12,24,30\rangle\ \text{(Maple)}}$$

$$=\underbrace{c_1e^{-10t}b_1+c_2e^{-5t}b_2+c_3e^{-4t}b_3}_{\text{transient}}+\underbrace{\begin{bmatrix}12\\24\\30\end{bmatrix}}_{\text{equilibrium soln}}$$

transient $\to 0$ within about $5\tau_3=1.25$

approaches 1% of initial values.

Nonhomogeneous 1st order linear systems, etc

Since nonhomogeneous we can let $x(0)=0$ to get a nonzero soln.

So $\quad x_1(0)=0,\ x_2(0)=0,\ x_3(0)=0.$

$x(0) = \langle x_1(0), x_2(0), x_3(0)\rangle = \langle 0,0,0\rangle$, $\quad y(0) = \langle c_1, c_2, c_3\rangle$

$$x(0) = B\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} 12 \\ 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} -12 \\ -24 \\ -30 \end{bmatrix} = \begin{bmatrix} -20 \\ 240 \\ -250 \end{bmatrix} \quad \text{(maple)}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{-20e^{-10t}\begin{bmatrix} 3/5 \\ -6/5 \\ 1 \end{bmatrix} + 240e^{-5t}\begin{bmatrix} 0 \\ -1/5 \\ 1 \end{bmatrix} - 250e^{-4t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{3 decaying modes (transient)}} + \underbrace{\begin{bmatrix} 12 \\ 24 \\ 30 \end{bmatrix}}_{\text{equilibrium soln (steady state)}}$$

$$\left.\begin{aligned} x_1 &= -12e^{-10t} + 12 \\ x_2 &= 24e^{-10t} - 48e^{-5t} + 24 \\ x_3 &= -20e^{-10t} + 240e^{-5t} - 250e^{-4t} + 30 \end{aligned}\right\} \text{scalar form of soln.}$$

Second order case decoupled DEs

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 200 \\ -1200 \\ 1000 \end{bmatrix} = \begin{bmatrix} -10y_1 + 200 \\ -5y_2 - 1200 \\ -4y_3 + 1000 \end{bmatrix}$$

negative eigenvalues lead to oscillations

$$\left.\begin{aligned} &y_1'' + 10y_1 = 200 && y_{1h} = c_1\cos\sqrt{10}t + c_2\sin\sqrt{10}t && y_{1p} = 20 \\ &y_2'' + 5y_2 = -1200 && y_{2h} = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t && y_{2p} = -240 \\ &y_3'' + 4y_3 = 1000 && y_{3h} = c_5\cos 2t + c_6\sin 2t && y_{3p} = 250 \end{aligned}\right\} \text{same as first order case}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = B\begin{bmatrix} c_1\cos\sqrt{10}t + c_2\sin\sqrt{10}t + 20 \\ c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t - 240 \\ c_5\cos 2t + c_6\sin 2t + 250 \end{bmatrix} = \underbrace{B\begin{bmatrix} \cdots \end{bmatrix}}_{x_h} + \underbrace{\begin{bmatrix} 12 \\ 24 \\ 30 \end{bmatrix}}_{x_p} \quad \text{(same as before)}$$

$$= (c_1\cos\sqrt{10}t + c_2\sin\sqrt{10}t)\,b_1 + (c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t)\,b_2 + (c_5\cos 2t + c_6\sin 2t)\,b_3 + x_p$$

3 oscillating modes: $\quad \omega = \sqrt{10}, \sqrt{5}, \sqrt{4} \approx 3.16,\ 2.24,\ 2$

$\quad T = \frac{2\pi}{\sqrt{10}}, \frac{2\pi}{\sqrt{5}}, \frac{2\pi}{2} \approx 1.99,\ 2.81,\ 3.14$

Initial conditions: $x(0) = \emptyset$, $x'(0) = \emptyset$ — just for the sake of an example.

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_3 \\ c_5 \end{bmatrix} + \begin{bmatrix} 12 \\ 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_3 \\ c_5 \end{bmatrix} = -B^{-1} \begin{bmatrix} 12 \\ 24 \\ 30 \end{bmatrix} = \begin{bmatrix} -20 \\ 240 \\ -250 \end{bmatrix}$$ same as before

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = B \begin{bmatrix} -c_1\sqrt{10}\sin\sqrt{10}t + c_2\sqrt{10}\cos\sqrt{10}t \\ -c_3\sqrt{5}\sin\sqrt{5}t + c_4\sqrt{5}\cos\sqrt{5}t \\ -c_5\cdot 2\sin 2t + c_6\cdot 2\cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{bmatrix} = B \begin{bmatrix} \sqrt{10}c_2 \\ \sqrt{5}c_4 \\ 2c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} \sqrt{10}c_2 \\ \sqrt{5}c_4 \\ 2c_6 \end{bmatrix} = B^{-1}\emptyset = \emptyset \to c_2 = c_4 = c_6 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -20\cos\sqrt{10}t \begin{bmatrix} 3/5 \\ -6/5 \\ 1 \end{bmatrix} + 240\cos\sqrt{5}t \begin{bmatrix} 0 \\ -1/5 \\ 1 \end{bmatrix} - 250\cos 2t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 12 \\ 24 \\ 30 \end{bmatrix}}_{\text{new equilibrium}}$$

$$\left.\begin{aligned} x_1 &= -12\cos\sqrt{10}t + 12 \\ x_2 &= 24\cos\sqrt{10}t - 48\cos\sqrt{5}t + 24 \\ x_3 &= -20\cos\sqrt{10}t + \underbrace{240\cos\sqrt{5}t - 250\cos 2t}_{} + 30 \end{aligned}\right\} \text{scalar form of soln}$$

much smaller, dominated by → beating $\omega_2 \approx 2.24$, $\omega_3 = 2$ relatively close, coefficients nearly opposite

plot shows this clearly

$$T_{beat} = \frac{2\pi}{2.24-2} \approx 26.2 \gg T_1, T_2$$

(see dramatic Maple plot)

Nonhomogeneous case: $\vec{x}\,' = A\vec{x} + \vec{F}$

$$x_1' = x_2 + 2 \;,\; x_1(0) = 1 \qquad x_2' = x_1 \;,\; x_2(0) = 0 \quad \Rightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 + 2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1x_2 + 2 \\ 1x_1 + 0x_2 + 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{F}}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Diagonalization $\quad \lambda = 1, -1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \;(\vec{b}_1\ \vec{b}_2) \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

change of basis

$$\vec{x} = y_1\vec{b}_1 + y_2\vec{b}_2 = B\vec{y} \;,\; \vec{y} = B^{-1}\vec{x}$$

transform DE:

$$\vec{x}\,' = A\vec{x} + \vec{F} \longrightarrow \quad (B\vec{y})' = A(B\vec{y}) + \vec{F} \rightarrow B^{-1}(B\vec{y}\,') = B^{-1}AB\vec{y} + B^{-1}\vec{F}$$

$$\vec{y}\,' = A_B\vec{y} + B^{-1}\vec{F} \qquad \text{new components of } \vec{F}: \; B^{-1}\vec{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} y_1 + 1 \\ -y_2 - 1 \end{pmatrix} \qquad \text{solve decoupled DEs:}$$

$$y_1' = y_1 + 1 \;,\; y_{1h} = c_1 e^{t} \quad y_{1p} = c_3 \rightarrow \underset{0}{(c_3)'} = c_3 + 1 \rightarrow c_3 = -1$$

$$y_2' = -y_2 + 1 \quad y_{2h} = c_2 e^{-t} \quad y_{2p} = c_4 \qquad (c_4)' = -c_4 - 1 \rightarrow c_4 = -1$$

$$\vec{y}_h = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{bmatrix},\; \vec{y}_p = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\; \vec{y} = \vec{y}_h + \vec{y}_p = \begin{bmatrix} c_1 e^{t} - 1 \\ c_2 e^{-t} - 1 \end{bmatrix}$$

backsub:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{t} - 1 \\ c_2 e^{-t} - 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \underbrace{c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{x}_p}$$

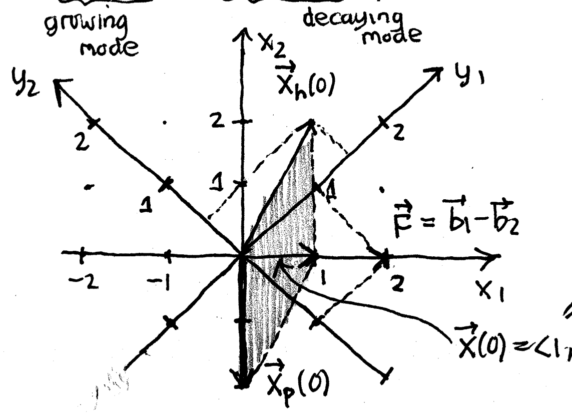
initial conditions:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{B}\begin{bmatrix} c_1 - 1 \\ c_2 - 1 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 - 1 \\ c_2 - 1 \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$c_1 - 1 = 1/2 \rightarrow c_1 = 3/2$$
$$c_2 - 1 = -1/2 \rightarrow c_2 = 1/2$$

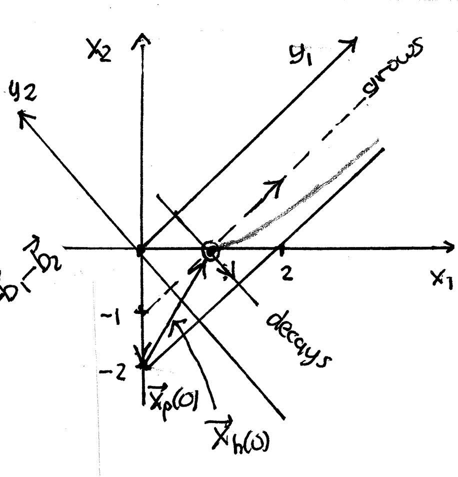
backsub:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\frac{3}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{growing mode}} + \underbrace{\frac{1}{2}e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{decaying mode}} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/2\,e^{t} - 1/2\,e^{-t} \\ 3/2\,e^{t} + 1/2\,e^{-t} - 2 \end{bmatrix} \qquad \vec{x}_h(0) = \begin{bmatrix} 3/2 - 1/2 \\ 3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$= \tfrac{1}{2}\vec{b}_1 - \tfrac{1}{2}\vec{b}_2 \qquad = \tfrac{3}{2}\vec{b}_1 + \tfrac{1}{2}\vec{b}_2 \qquad = -\vec{b}_1 - \vec{b}_2$$

$$\vec{x}(0) = \langle 1, 0\rangle = \vec{x}_h(0) + \vec{x}_p(0)$$



2nd order Nonhomogeneous case : $\vec{x}'' = A\vec{x} + \vec{F}$

$x_1'' = x_2 + 2 \qquad x_1(0) = 1,\ x_1'(0) = 0$

$x_2'' = x_1 \qquad x_2(0) = 0,\ x_2'(0) = 1$

$$\Rightarrow \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} x_2 + 2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1x_2 + 2 \\ 1x_1 + 0x_2 + 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{F}}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

diagonalization: $\lambda = 1, -1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad B^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad A_B = B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\qquad\qquad\quad \vec{b}_1 \ \vec{b}_2$

change of basis:

$$\vec{x} = y_1\vec{b}_1 + y_2\vec{b}_2 = B\vec{y}, \quad \vec{y} = B^{-1}\vec{x}$$

transform DE:

$$\vec{x}'' = A\vec{x} + \vec{F} \rightarrow (B\vec{y})'' = A(B\vec{y}) + \vec{F} \rightarrow B^{-1}(B\vec{y}'') = B^{-1}AB\vec{y} + B^{-1}\vec{F}$$

$$\vec{y}'' = A_B\vec{y} + B^{-1}\vec{F} \qquad \text{new components of } \vec{F}: \ B^{-1}\vec{F} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} y_1 + 1 \\ -y_2 - 1 \end{bmatrix}$$

solve decouped 2nd order DEs:

$$\boxed{\begin{matrix} y_1'' = y_1 + 1 \\ y_2'' = -y_2 - 1 \end{matrix}} \rightarrow \begin{matrix} y_1'' - y_1 = 1 & y_{1h} = e^{rt} \rightarrow r^2 - 1 = 0,\ r = \pm 1,\ e^{rt} = e^{\pm t}, & y_{1h} = c_1e^t + c_2e^{-t} \\ y_2'' + y_2 = -1 & y_{2h} = e^{rt} \rightarrow r^2 + 1 = 0,\ r = \pm i,\ e^{rt} = e^{\pm it} = \cos t \pm i\sin t & \end{matrix}$$

(unknowns on LHS ← nonhom driving terms)

$y_{2h} = c_3\cos t + c_4\sin t$

$y_{1p} = c_5 \qquad (c_5)'' - c_5 = 1 \rightarrow c_5 = -1$

$y_{2p} = c_6 \qquad \underset{0}{(c_6)''} + c_6 = -1 \rightarrow c_6 = -1$

$y_1 = y_{1h} + y_{1p} = c_1e^t + c_2e^{-t} - 1$

$y_2 = y_{2h} + y_{2p} = c_3\cos t + c_4\sin t - 1$

backsub:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^t + c_2e^{-t} - 1 \\ c_3\cos t + c_4\sin t - 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^t + c_2e^{-t} \\ c_3\cos t + c_4\sin t \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \underbrace{(c_1e^t + c_2e^{-t})\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3\cos t + c_4\sin t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{x}_p}$$

initial conditions:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^t - c_2e^{-t} \\ -c_3\sin t + c_4\cos t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 + c_2 - 1 \\ c_3 - 1 \end{bmatrix} \qquad \begin{bmatrix} c_1 + c_2 - 1 \\ c_3 - 1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$c_1 + c_2 - 1 = 1/2$

$c_3 - 1 = -1/2 \rightarrow c_3 = 1 - 1/2 = 1/2$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 - c_2 \\ c_4 \end{bmatrix} \qquad \begin{bmatrix} c_1 - c_2 \\ c_4 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$c_1 - c_2 = 1/2$

$c_4 = 1/2 \rightarrow c_4 = 1/2$

$c_1 + c_2 = 1 + 1/2 = 3/2$

$c_1 - c_2 = 1/2$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}, \ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/4 \\ 2/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{(e^t + \tfrac{1}{2}e^{-t})}_{\text{growth along } \vec{b}_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{(\tfrac{1}{2}\cos t + \tfrac{1}{2}\sin t)}_{\text{oscillation along } \vec{b}_2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} e^t + \tfrac{1}{2}e^{-t} - \tfrac{1}{2}\cos t - \tfrac{1}{2}\sin t \\ e^t + \tfrac{1}{2}e^{-t} + \tfrac{1}{2}\cos t + \tfrac{1}{2}\sin t - 2 \end{bmatrix}$$