

2.3b complex eigenvalues: example

①

$$\frac{dx_1}{dt} = x_1 + 2x_2 \quad \rightarrow \quad \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\frac{dx_2}{dt} = -4x_1 - 3x_2$$

$$x_1(0) = 1, \quad x_2(0) = 2$$

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ -4 & -3-\lambda \end{vmatrix} = -(3+\lambda)(1-\lambda) + 8 = (\lambda-1)(\lambda+3) + 8$$

$$= \lambda^2 + 2\lambda - 3 + 8 = \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-4 \cdot 5}}{2} = -1 \pm \sqrt{-5} = -1 \pm 2i$$

$$\lambda = -1 + 2i: \quad A + (1-2i)I = \begin{bmatrix} 1+1-2i & 2 \\ -4 & -3+1-2i \end{bmatrix} = \begin{bmatrix} 2-2i & 2 \\ -4 & -2-2i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2}(1+i) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = t, \quad x_1 = -\frac{1}{2}(1+i)t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(1+i)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

$$\lambda = -1 + 2i, \quad -1 - 2i$$

$$B = \begin{bmatrix} -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ 1 & 1 \end{bmatrix}$$

$$A_B = B^{-1}AB = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix} \quad b_1 \rightarrow b_2 = \bar{b}_1$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (-1+2i)y_1 \\ (-1-2i)y_2 \end{bmatrix} \quad \begin{matrix} y_1' = (-1+2i)y_1 & y_1 = e_1 e^{(-1+2i)t} \\ y_2' = (-1-2i)y_2 & y_2 = e_2 e^{(-1-2i)t} \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 b_1 + y_2 b_2 = \underbrace{e_1 e^{(-1+2i)t} \begin{bmatrix} -\frac{1-i}{2} \\ 1 \end{bmatrix}}_{Z = X_1 + iX_2} + \underbrace{e_2 e^{(-1-2i)t} \begin{bmatrix} -\frac{1+i}{2} \\ 1 \end{bmatrix}}_{\bar{Z} = X_1 - iX_2}$$

$$= e_1 (X_1 + iX_2) + e_2 (X_1 - iX_2) = \underbrace{(e_1 + e_2)}_{c_1} X_1 + \underbrace{i(e_1 - e_2)}_{c_2} X_2$$

$$= c_1 X_1 + c_2 X_2$$

← must be real for real solns

so the real and imaginary parts of the complex vector function solns Z & \bar{Z} are a real basis of the soln space.

we just need to evaluate the Re & Im parts of Z to get this new basis.

7.3b) complex eigenvalues: example

(2)

$$Z = e^{(1+2i)t} \begin{bmatrix} -\frac{1-i}{2} \\ 1 \end{bmatrix} = e^{-t} (\cos 2t + i \sin 2t) \begin{bmatrix} -\frac{1-i}{2} \\ 1 \end{bmatrix} \leftarrow \text{one complex multiplication required}$$

$$e^{-t} \frac{e^{2it}}{\cos 2t + i \sin 2t} = e^{-t} \begin{bmatrix} \frac{1}{2}(-\cos 2t + i \sin 2t) + \frac{i}{2}(-\cos 2t - \sin 2t) \\ \cos 2t + i \sin 2t \end{bmatrix}$$

$$= e^{-t} \left[\frac{1}{2}(-\cos 2t + i \sin 2t) \right] + i e^{-t} \left[\frac{1}{2}(-\cos 2t - \sin 2t) \right]$$

$X_1 \qquad X_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 X_1 + c_2 X_2 = c_1 \begin{bmatrix} \frac{1}{2}(-\cos 2t + i \sin 2t) \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2}(-\cos 2t - \sin 2t) \\ \sin 2t \end{bmatrix}$$

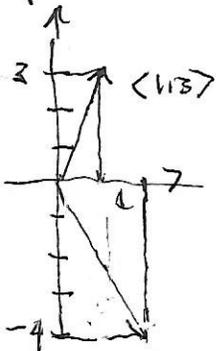
$$= \begin{bmatrix} -\frac{1}{2}(c_1 + c_2) \cos 2t + \frac{1}{2}(c_1 - c_2) \sin 2t \\ c_1 \cos 2t + c_2 \sin 2t \end{bmatrix} \text{ general soln.}$$

initial conditions

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(c_1 + c_2) \\ c_1 \end{bmatrix} \rightarrow c_1 = 2 \rightarrow \begin{aligned} -\frac{1}{2}(c_1 + c_2) &= 1 \rightarrow c_2 = -2 - c_1 = -2 - 2 = -4 \\ \frac{1}{2}(c_1 - c_2) &= \frac{1}{2}(2 - (-4)) = 3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos 2t + 3 \sin 2t) \\ e^{-t}(2 \cos 2t - 4 \sin 2t) \end{bmatrix} = \begin{bmatrix} \sqrt{10} e^{-t} \cos(2t - \arctan 3) \\ 2\sqrt{5} e^{-t} \cos(2t + \arctan 2) \end{bmatrix}$$

amplitude and phase shift?



$$A_1 = \sqrt{1+9} = \sqrt{10}$$

$$\delta_1 = \arctan 3$$

$$A_2 = \sqrt{2^2 + 4^2} = 2\sqrt{5}$$

$$\delta_2 = -\arctan 2$$

envelope functions

6.1 Example 6

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$$

$$0 = |A - \lambda I| \stackrel{\text{Maple}}{=} \dots = -\lambda(\lambda-1)(\lambda-3) \rightarrow \lambda = 0, 1, 3 \quad (2)$$

$$\lambda_1 = 0: A - 0I = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & -15 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x_1, x_2, x_3
LLF

$$x_3 = t: \begin{matrix} x_1 = 0 \\ x_1 + \frac{1}{3}x_3 = 0 \\ 0 = 0 \end{matrix} \rightarrow \begin{matrix} x_1 = -\frac{1}{3}x_3 \\ = -\frac{1}{3}t \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix} \rightarrow b_1 = \begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\lambda_2 = 1: A - I = \begin{bmatrix} 3-1 & 0 & 0 \\ -4 & 6-1 & 2 \\ 16 & -15 & -5-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & -15 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x_1, x_2, x_3
LLF

$$x_3 = t: \begin{matrix} x_1 = 0 \\ x_2 + \frac{2}{5}x_3 = 0 \\ 0 = 0 \end{matrix} \rightarrow x_2 = -\frac{2}{5}t \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/5t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix} \rightarrow b_2 = \begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

$$\lambda_3 = 3: A - 3I = \begin{bmatrix} 3-3 & 0 & 0 \\ -4 & 6-3 & 2 \\ 16 & -15 & -5-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 16 & -15 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 0 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-16 + 12 + 8$

$$x_3 = t: \begin{matrix} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{matrix} \rightarrow x_1 = \frac{1}{2}t \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \rightarrow b_3 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$B = \langle b_1, b_2, b_3 \rangle = \begin{bmatrix} 0 & 0 & 1/2 \\ -1/3 & -2/5 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ 3 & 5 & 2 \end{bmatrix}$$

Maple \rightarrow integer choice

either choice is a basis of \mathbb{R}^3 of eigenvectors of A or an "eigenbasis."

In these simple exercises, the row reduction is easy to do by hand BUT you can use Maple, to be sure of the result.

Then $x = By$, $y = B^{-1}x$ is the decoupling change of variables we need to solve $\frac{dx}{dt} = Ax$.