

7.3b) Linear homogeneous 1st order DEs: complex eigenvalues ①

$$x' = \underbrace{A}_{n \times n} x$$

When the coefficient matrix A is diagonalizable over the complex numbers, we can use the same approach as in the real case EXCEPT we have the option to choose the real and imaginary parts to replace a complex eigenvector solution and its complex conjugate partner (although we can also just push through using complex coordinates). The complex eigenvectors turn out to contain essential information about the relative initial amplitudes and phase shifts of the various real variables in a single "complex mode".

EXAMPLE:  $A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \xrightarrow{\text{Maple}} B = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ 1 & 1 \end{bmatrix}$   $B^{-1} = \begin{bmatrix} i & \frac{1}{2} + \frac{i}{2} \\ -i & \frac{1}{2} - \frac{i}{2} \end{bmatrix} \xrightarrow{\text{Maple}}$

 $\lambda = -1+2i, -1-2i$ 
 $A_B = B^{-1}AB = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{\substack{x = By \\ y = B^{-1}x}} y' = A_B y$$

$$\frac{dy_1}{dt} = (-1+2i)y_1 \rightarrow y_1 = e_1 e^{(-1+2i)t}$$

$$\frac{dy_2}{dt} = (-1-2i)y_2 \rightarrow y_2 = e_2 e^{(-1-2i)t}$$

decoupled DEs

eigenvalue  
exponential  
solutions

along eigenvector  
directions

$$\begin{cases} \frac{dx_1}{dt} = x_1 + 2x_2 \\ \frac{dx_2}{dt} = -4x_1 - 3x_2 \end{cases}$$

scalar  
DEs

general solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{(-1+2i)t} \\ c_2 e^{(-1-2i)t} \end{bmatrix} = e_1 e^{(-1+2i)t} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + e_2 e^{(-1-2i)t} \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

(unmultiplied) matrix form

vector form

(linear combination of complex conjugate modes)

For real solutions  $\tilde{x} = \underbrace{\overline{e}_1 e^{(-1-2i)t}}_{\downarrow b_1} + \underbrace{\overline{e}_2 e^{(-1+2i)t}}_{\downarrow b_2} \underbrace{\overline{b}_2}_{b_1} = x$

 $= c_2 \quad \Rightarrow \quad c_2 = \overline{c}_1$

the coefficients must also be complex conjugates ("c, c.")

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overline{c}_1 e^{(-1+2i)t} \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix} + \text{c.c.} = \operatorname{Re} \left( 2\overline{c}_1 e^{(-1+2i)t} \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix} \right)$$

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## Linear homogeneous 1st order DEs: complex eigenvalues

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We have two options:

- 1) continue with the complex vector soln basis pair or
- 2) switch to the real & imaginary parts for an explicitly real basis.

Complex coordinate approach:

$$\text{Initial conditions: } x_1(0) = 1, x_2(0) = 2 \rightarrow \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} i & \frac{1+i}{2} \\ -i & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} i + (1+i) \\ -i + (1-i) \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1-2i \end{bmatrix}$$

$$2e_1 b_1 = (1+2i) \begin{bmatrix} 1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -1+2 \\ 2+4i \end{bmatrix} = \begin{bmatrix} 1-3i \\ 2+4i \end{bmatrix} = \begin{bmatrix} \sqrt{10} e^{-i\arctan 3} \\ 2\sqrt{5} e^{i\arctan 2} \end{bmatrix} \xleftarrow{\text{compare}}$$

$$2e_1 e^{\lambda_1 t} b_1 = e^{-t} (\cos 2t + i \sin 2t) \begin{bmatrix} 1-3i \\ 2+4i \end{bmatrix} = e^{-t} \left[ \begin{array}{l} \cos 2t + 3 \sin 2t + i(-3 \cos 2t + \sin 2t) \\ 2 \cos 2t - 4 \sin 2t + i(8 \cos 2t + 2 \sin 2t) \end{array} \right]$$

$$\text{Re}(2e_1 e^{\lambda_1 t} b_1) = \underbrace{e^{-t} \left[ \begin{array}{l} \cos 2t + 3 \sin 2t \\ 2 \cos 2t - 4 \sin 2t \end{array} \right]}_{\text{Solution functions}} = \underbrace{e^{-t} \left[ \begin{array}{l} \sqrt{10} \cos(2t - \arctan 3) \\ 2\sqrt{5} \cos(2t + \arctan 2) \end{array} \right]}_{\text{needed for interpretation}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In general we can pass to a real basis by expressing the complex formula in terms of the real and imaginary parts of all the factors:

$\vec{z} = e^{\lambda_1 t} b_1 = \vec{x}_1 + i \vec{x}_2$ ,  $\bar{\vec{z}} = \vec{x}_1 - i \vec{x}_2$  are a complex basis of the soln space. Their real and imaginary parts  $\{\vec{x}_1, \vec{x}_2\}$  are real basis solns.

Let  $2e_1 = c_1 - i c_2$ , then

$$x = \text{Re}(2e_1 e^{\lambda_1 t} b_1) = \text{Re}((c_1 - i c_2)(\vec{x}_1 + i \vec{x}_2)) = \underbrace{c_1 \vec{x}_1 + c_2 \vec{x}_2}_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

Calculate:

$$\begin{aligned} e^{\lambda_1 t} b_1 &= e^{-t} (\cos 2t + i \sin 2t) \begin{bmatrix} -\frac{1-i}{2} \\ 1 \end{bmatrix} = e^{-t} \left[ \begin{array}{l} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t + i(-\frac{1}{2} \cos 2t - \frac{1}{2} \sin 2t) \\ \cos 2t + i \sin 2t \end{array} \right] \\ &= \underbrace{e^{-t} \left[ \begin{array}{l} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ \cos 2t \end{array} \right]}_{\vec{x}_1} + i \underbrace{e^{-t} \left[ \begin{array}{l} -\frac{1}{2} \cos 2t - \frac{1}{2} \sin 2t \\ \sin 2t \end{array} \right]}_{\vec{x}_2} \end{aligned}$$

$$\begin{aligned} \vec{x}_1 &= e^{-t} \left[ \begin{array}{l} \frac{1}{2\sqrt{2}} \cos(2t - 3\pi/4) \\ 1 \cos 2t \end{array} \right] \\ &= \begin{bmatrix} \frac{1}{2\sqrt{2}} \cos(2t - 3\pi/4) \\ 1 \cos 2t \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \vec{x}_2 &= e^{-t} \left[ \begin{array}{l} \frac{1}{2\sqrt{2}} \cos(2t + 3\pi/4) \\ 1 \cos 2t \end{array} \right] \\ &= \begin{bmatrix} \frac{1}{2\sqrt{2}} \cos(2t + 3\pi/4) \\ 1 \cos 2t \end{bmatrix} \end{aligned}$$

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# Linear homogeneous 1st order DEs: complex eigenvalues

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Both these modes are of the form

$$e^{-t} \begin{bmatrix} A_1 \cos(2t - \delta_1) \\ A_2 \cos(2t - \delta_2) \end{bmatrix} \text{ with } \frac{A_1}{A_2} = \frac{1}{2\sqrt{2}}, \quad \delta_1 - \delta_2 = \frac{3\pi}{4} - 0 = \frac{3\pi}{4} \leftarrow \\ \delta_1 - \delta_2 = -\frac{3\pi}{4} - \frac{\pi}{2} = -\frac{5\pi}{4} = \frac{3\pi}{4} - 2\pi$$

$$\text{while } b_1 = \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} e^{-i\frac{3\pi}{4}/4} \\ 1 e^0 \end{bmatrix}$$

The complex eigenvector sets the ratios of the initial amplitudes of the sinusoidal factor while also setting their phase difference. Thus the physical information about the two independent real modes is determined by the polar form of the complex eigenvector.

However, we do not need to do all this complex arithmetic. We can simply use the real basis  $\{x_1, x_2\}$  to express the general soln:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\frac{1}{2} \cos 2t - \frac{1}{2} \sin 2t \\ \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(c_1 + c_2) \\ c_1 \end{bmatrix} \begin{array}{l} -\frac{1}{2}(c_1 + c_2) = 1 \rightarrow c_2 = -2 - c_1 \\ c_1 = 2 \end{array} = -4$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-t} \begin{bmatrix} 2(-\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t) - 4(-\frac{1}{2} \cos 2t - \frac{1}{2} \sin 2t) \\ 2 \cos 2t - 4 \sin 2t \end{bmatrix}$$

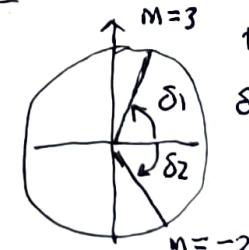
$$= e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2 \cos 2t - 4 \sin 2t \end{bmatrix} = e^{-t} \begin{bmatrix} \frac{1}{2\sqrt{2}} \cos(2t - \arctan 3) \\ 1 \cos(2t + \arctan 2) \end{bmatrix}$$

phase difference of  $\frac{3\pi}{4}$   
is hidden!

$$\arctan 3 - (-\arctan 2) = \Delta \delta \\ = \underbrace{\arctan 3}_A + \underbrace{\arctan 2}_B$$

$$\begin{aligned} \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{3+2}{1-3 \cdot 2} \\ &= \frac{5}{-5} = -1 \\ \tan(A+B) &= -1 \end{aligned}$$

sticking with real functions requires complicated trig formulas — easy with complex exponentials!



$$\tan(\delta_1 - \delta_2) = -1 \\ \delta_1 - \delta_2 = \frac{3\pi}{4}$$

7.3b

## Linear homogeneous 1st order DEs: complex eigenvalues ④

What's the take away?

For a complex eigenvalue pair  $\lambda \pm i\delta$ ,  $b \pm$

just evaluate  $e^{\lambda t} b_+ = \vec{x}_1 + i\vec{x}_2$

Then the general soln is  $x = c_1 \vec{x}_1 + c_2 \vec{x}_2$ .

Then initial conditions set the coefficients of the resulting pair of real modes.

The polar form of the complex eigenvector determines the relative initial amplitudes and relative phase shifts of the sinusoidal factors of the individual scalar variables in these modes.

The initial conditions only set the overall amplitude scale and the overall phase of the mode.

$$\left. \begin{array}{l} \text{Suppose } \lambda_1 = k + i\omega \\ b_1 = \begin{bmatrix} A_1 e^{i\delta_1} \\ A_2 e^{i\delta_2} \end{bmatrix} \\ 2c_1 = c_1 - i c_2 = A e^{-i\delta} \end{array} \right\}$$

$$\begin{aligned} 2c_1 e^{\lambda_1 t} b_1 &= e^{kt} A e^{-i\delta} e^{i\omega t} \begin{bmatrix} A_1 e^{i\delta_1} \\ A_2 e^{i\delta_2} \end{bmatrix} \\ &= e^{kt} \begin{bmatrix} AA_1 e^{i(\omega t - \delta + \delta_1)} \\ AA_2 e^{i(\omega t - \delta + \delta_2)} \end{bmatrix} \end{aligned}$$

↑  
overall amplitudes      ↑  
phase difference  
set by initial conditions but relative values  
fixed by DE system

The matrix  $A$  forces the scalar variables to evolve always in the same amplitude ratio and phase difference as the overall scale grows or decays exponentially.

7.3b) Linear homogeneous 1st order DEs: complex eigenvalues (5)

Okay, let's do another example without all the explanatory words getting in the way.

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - 5x_2 \\ \frac{dx_2}{dt} &= x_1 - 3x_2 \end{aligned} \quad \left. \begin{array}{l} \text{Maple} \\ \downarrow \end{array} \right\} \begin{aligned} x' &= \begin{bmatrix} 1-5 \\ 1-3 \end{bmatrix} x \rightarrow B = \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix} \\ A & \lambda = -1+i, -1-i \end{aligned}$$

$$e^{\lambda t} b_i = e^{(-1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = e^{-t} (\cos t + i \sin t) \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} 2 \cos t - \sin t + i(\cos t + 2 \sin t) \\ \cos t + i \sin t \end{bmatrix}$$

$$x_1 = e^{-t} \begin{bmatrix} 2 \cos t - \sin t \\ \cos t \end{bmatrix}, x_2 = e^{-t} \begin{bmatrix} \cos t + 2 \sin t \\ \sin t \end{bmatrix} \quad \text{real basis of sdn space}$$

$$\text{Gen soln: } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 x_1 + c_2 x_2 = c_1 e^{-t} \begin{bmatrix} 2 \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \cos t + 2 \sin t \\ \sin t \end{bmatrix}$$

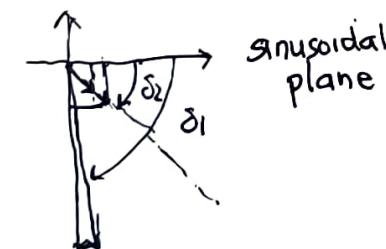
$$\text{initial conditions: } x_1(0) = 1, x_2(0) = 2 \rightarrow \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{array}{l} c_2 = 1 - 2c_1 = -3 \\ c_1 = 2 \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2e^{-t} \begin{bmatrix} 2 \cos t - \sin t \\ \cos t \end{bmatrix} - 3e^{-t} \begin{bmatrix} \cos t + 2 \sin t \\ \sin t \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \cos t - 8 \sin t \\ 2 \cos t - 3 \sin t \end{bmatrix} \quad \text{sln of IVP}$$

$$= e^{-t} \begin{bmatrix} \sqrt{65} \cos(t + \arctan \theta) \\ \sqrt{13} \cos(t + \arctan 3/2) \end{bmatrix} \quad \text{interpretation}$$



$$\langle 1, -8 \rangle = \sqrt{65} \langle \cos \delta_1, \sin \delta_1 \rangle$$

$$\langle 2, -3 \rangle = \sqrt{13} \langle \cos \delta_2, \sin \delta_2 \rangle$$

$$\begin{aligned} \sqrt{65}/\sqrt{13} &= \sqrt{5}, \quad \delta_1 - \delta_2 = -\arctan \theta - (-\arctan 3/2) \\ &= \arctan 3/2 - \arctan \theta \\ &= -\arctan(1/2) \end{aligned}$$

Not so bad, eh?

$$\boxed{\begin{array}{l} \text{complex eigenvector:} \\ b_2 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{5} e^{i \arctan \theta} \\ 1 e^0 \end{bmatrix} \\ \leftarrow \text{reverse sign for phase shift.} \end{array}}$$