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1st order linear homogeneous DE systems: real eigenvalues ①

When we can diagonalize a matrix with real eigenvalues and eigenvectors, we get real vector exponential functions for the solutions.

$$\begin{aligned} n=1: \quad \dot{x} = kx \rightarrow x = e^{kt} C & \quad \text{scalar variable} \\ n>1: \quad \dot{\vec{x}} = A\vec{x} \rightarrow \vec{x} = \sum e^{\lambda t} \vec{v} & \quad \text{vector variable} \quad (\text{arrows for emphasis}) \\ & \quad \uparrow \quad \downarrow \\ & \quad \text{eigenvalues and eigenvectors} \end{aligned}$$

This is best explained with an explicit example:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \xrightarrow{\text{Maple}} B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, B^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, A_B = B^{-1} A B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\lambda = 5, -1$$

matrix form	scalar form
DEs $\dot{x} = Ax$: $\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\frac{dx_1}{dt} = x_1 + 4x_2$ $\frac{dx_2}{dt} = 2x_1 + 3x_2$
Initial conditions: $x(0) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$	$x_1(0) = -2, x_2(0) = 4$
Change of variable: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	$x_1 = y_1 - 2y_2$ $x_2 = y_1 + y_2$
$x = By$ $y = B^{-1}x$	$y_1 = \frac{1}{3}(x_1 + 2x_2)$ $y_2 = \frac{1}{3}(-x_1 + x_2)$

Decouple the DEs by transforming the DE:

$$\frac{d\vec{x}}{dt} = A\vec{x} \rightarrow \underbrace{\frac{d}{dt}(By)}_{= B \frac{dy}{dt}} = A(By) \rightarrow \frac{dy}{dt} = \underbrace{B^{-1}AB}_{A_B} y$$

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decoupled:

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5y_1 \\ -y_2 \end{bmatrix} \rightarrow y_1' = 5y_1 \rightarrow y_1 = c_1 e^{5t}$$

$$\rightarrow y_2' = -y_2 \rightarrow y_2 = c_2 e^{-t}$$

return to original variables:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{5t} \\ c_2 e^{-t} \end{bmatrix} = c_1 e^{5t} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\substack{\text{growing mode} \\ X_1}} + c_2 e^{-t} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\substack{\text{decaying mode} \\ X_2}}$$

vector form

matrix (product) form

The general soln is an arbitrary linear combination of the basis solns: $\{e^{21t}\vec{b}_1, e^{22t}\vec{b}_2\}: \vec{x} = c_1 e^{21t}\vec{b}_1 + c_2 e^{22t}\vec{b}_2$.

In our case we have a growing mode with $k_1 = 5, \tau_1 = 1/5$ and a decaying mode with $k_2 = -1, \tau_2 = 1$ which decays on a much longer timescale.

For each mode separately, all variables undergo exponential behavior with the same rate factor but with different initial values whose ratios are set by the ratios of the eigenvector components (which are only defined up to an overall scale factor). The initial conditions merely set the coefficients of each mode and therefore the actual initial values of each variable.

Solving the initial conditions is just a change of coordinates to determine those coefficients, which are the initial values of the new coordinates.

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -2+8 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

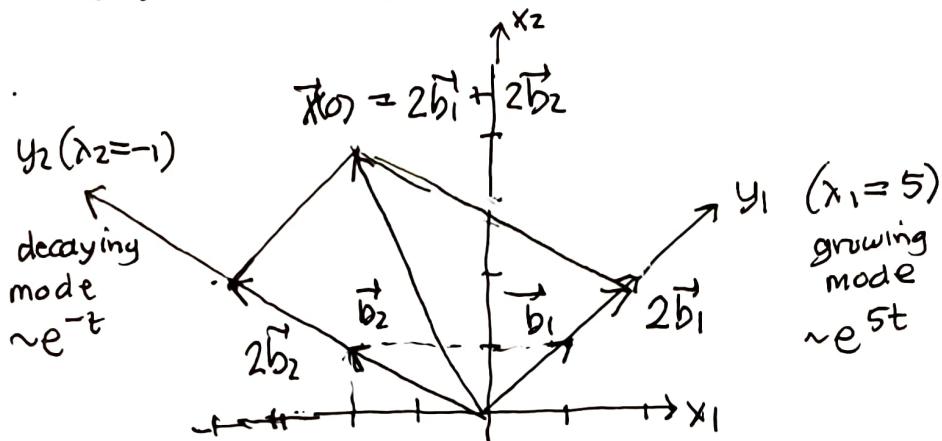
$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{vector form}} = \begin{bmatrix} 2e^{5t} - 4e^{-t} \\ 2e^{5t} + 2e^{-t} \end{bmatrix} \rightarrow \underbrace{\begin{aligned} x_1 &= 2e^{5t} - 4e^{-t} \\ x_2 &= 2e^{5t} + 2e^{-t} \end{aligned}}_{\text{scalar form}}$$

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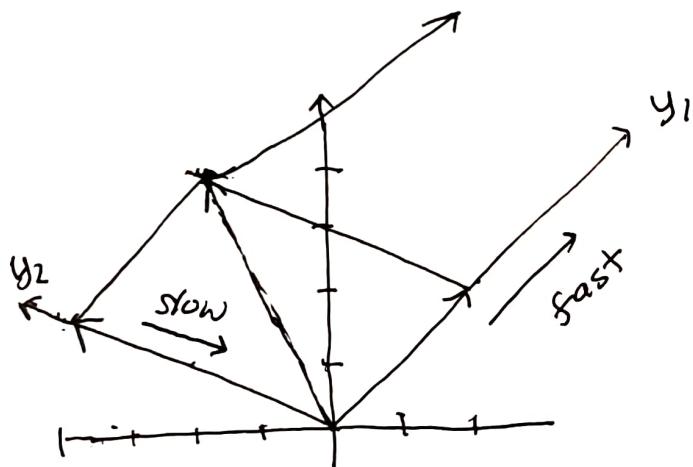
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However, the widely different time scales for the two modes means that the longer time scale dominates



The initial conditions set the initial values of each mode.



This is growing 5 times faster than it's decaying so in the time $t=0..5\tau_2=5$ it takes to decay by one factor of e , it grows by a factor $e^{5t} = e^{5(1)} \approx 148$ far out of our viewing window for these initial conditions.

When it has decayed to less than 1% of its initial value (the slow mode), it has grown (the fast mode) by

$$t = 0..5\tau_2 = 5 \rightarrow e^{5(5)} = e^{25} \approx 7 \times 10^{10}!!$$

Thus even though the y_1 axis is an asymptote for this solution curve, we never see it get close to that axis in realistic time scales.

In other words when we actually apply this math to real world problems we have to be aware of how all the relevant time scales in the problem affect the behavior.

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Summary. If A has a real eigenbasis, we can use it to decouple the corresponding DE system, solve it and return to the original variables, then solve the initial conditions by the same linear change of variables and obtain a linear combination of "real exponential modes":

$$\begin{array}{l} \text{d.e.s} \\ \downarrow \\ x' = Ax \xrightarrow{\substack{\text{end} \\ \text{result}}} \vec{x} = \sum_{i=1}^n c_i e^{\lambda_i t} \vec{b}_i \\ \text{eigen vector exponential soln basis functions: "modes"} \\ \downarrow \\ x = By \\ \downarrow \\ y' = B^{-1} A B y = A_B y \rightarrow y'_i = \lambda_i y_i \rightarrow y_i = c_i e^{\lambda_i t} \\ x = By = \sum_{i=1}^n y_i \vec{b}_i = \sum_{i=1}^n c_i e^{\lambda_i t} \vec{b}_i, \quad c_i = [B^{-1} x(0)]_i \end{array}$$

We can verify that it solves the DE system:

$$\begin{aligned} x' &= \frac{d}{dt} \left(\sum_{i=1}^n c_i e^{\lambda_i t} \vec{b}_i \right) = \sum_{i=1}^n c_i \lambda_i e^{\lambda_i t} \vec{b}_i && \boxed{\quad} \text{! agree} \\ Ax &= A \left(\sum_{i=1}^n c_i e^{\lambda_i t} \vec{b}_i \right) = \sum_{i=1}^n c_i e^{\lambda_i t} \underbrace{A \vec{b}_i}_{\lambda_i \vec{b}_i} \end{aligned}$$

When an eigenvalue is repeated (multiplicity $m > 1$), then the corresponding eigenspace all has the same exponential behavior so the terms in this sum will gather together.

For example: $\lambda = -1, -2, -2$

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= e^{-t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2e^{-2t} \underbrace{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}_{e^{-2t} \begin{pmatrix} 2(2) + 3(3) \\ 2(3) + 3(1) \\ 2(1) + 3(2) \end{pmatrix}} + 3e^{-2t} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-t} + 13e^{-2t} \\ 2e^{-t} + 9e^{-2t} \\ 3e^{-t} + 8e^{-2t} \end{pmatrix} \\ &\quad e^{-2t} \begin{pmatrix} 2(2) + 3(3) \\ 2(3) + 3(1) \\ 2(1) + 3(2) \end{pmatrix} = e^{-2t} \begin{pmatrix} 13 \\ 9 \\ 8 \end{pmatrix} \end{aligned}$$