Diagonalization geometry in the plane

The eigenvector diagonalization of a matrix brings with it a natural linear change of coordinates which is not only needed to solve initial conditions for the corresponding DE system, but also understand the behavior of the solutions to this system. The new coordinate grid provides the skeleton on which to hang the solution curves.

Visualizing this geometry is easiest to do in 2 dimensions (on paper, on screens), where "phase plots" of families of solutions are organized by the new coordinate grid, showing what behavior is possible.

The following step by step example is a template for the three offline homework problems.

0) Start with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1) Find eigenvalues: λ_1, λ_2 (choose $\lambda_1 \le \lambda_2$ so we all agree)

2) Find corresponding eigenvectors: $\vec{b}_1, \vec{b}_2$

3) Introduce basis changing matrix $B = \langle b_1 | b_2 \rangle$.

4) Confirm that $A_B = B^{-1} A B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

5) Make a diagram of the eigenvectors and new coordinate axes. Label them at the positive end by y_1, y_2 and also annotate with $\lambda_1 =$ value, $\lambda_2 =$ value to identify the corresponding eigenvalues.

6) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of some point $\langle x_1, x_2 \rangle = \vec{x}$ and graphically illustrate the decomposition of $\vec{x}$ into its two vector components along the coordinate axes, with parallelogram projection from the tip of $\vec{x}$ to those axes: $\vec{x} = y_1 \vec{b}_1 + y_2 \vec{b}_2$.
Recall: $\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x}$.

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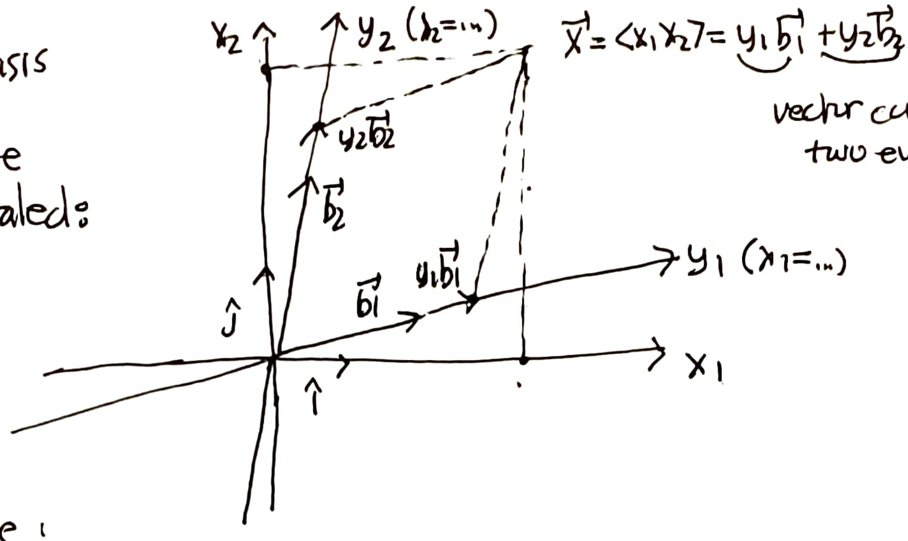
eigenbasis coord change illustrated:



$\vec{x} = \langle x_1, x_2\rangle = \underbrace{y_1\vec{b}_1}\,+\,\underbrace{y_2\vec{b}_2}$

vector components along two eigenvectors

Example:

0) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $\vec{x} = \langle -2, 4\rangle$

1) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda-1) - 8 = \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5$
$= (\lambda+1)(\lambda-5) = 0$
$\lambda = -1, 5$

2) $\lambda = -1$: $A + I = \begin{bmatrix} 1+1 & 4 \\ 2 & 3+1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 = -2t$, $x_2 = t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

L F

$\lambda = 5$: $A - 5I = \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 = t$, $x_2 = t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

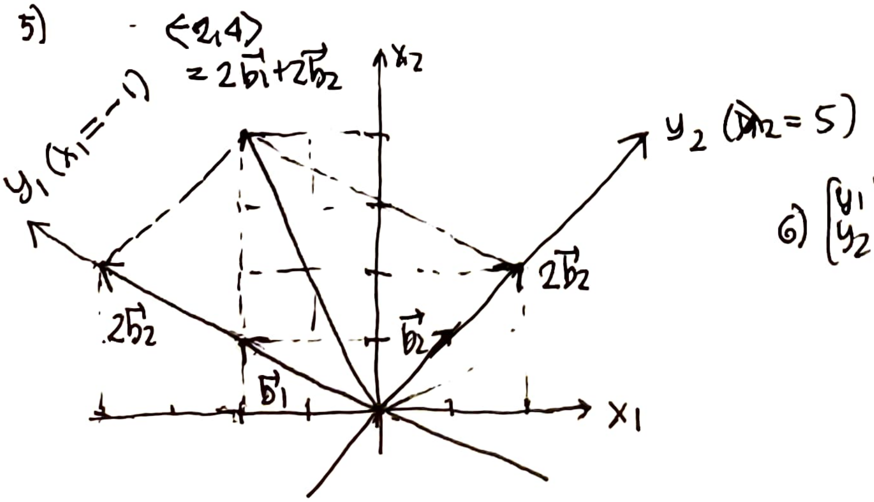
L F

3) $B = \langle b_1 | b_2\rangle = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$. $\quad B^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$

4) $A_B = B^{-1}AB = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$
$= \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$ ✓

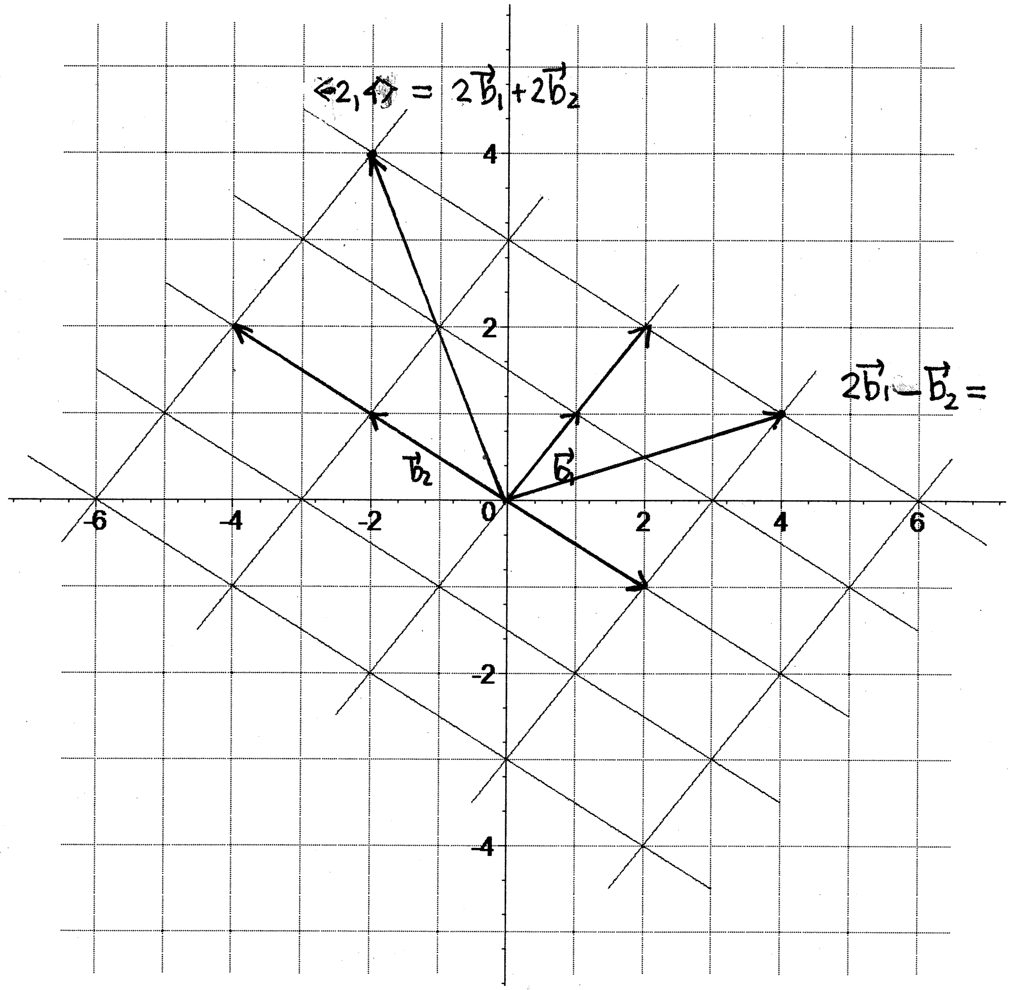
Yes, we can multiply matrices!

5)



6) $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2+4 \\ -2+8 \end{bmatrix}$
$= \frac{1}{3}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ✓

geometry of diagonalization



$\vec{b}_1 = \langle 1,1\rangle$, $\vec{b}_2 = \langle -2,1\rangle$ $\quad B = \begin{bmatrix}1&-2\\1&1\end{bmatrix}$ $\quad B^{-1} = \frac{1}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix}$

$\vec{x} = B\vec{y}$: $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1&-2\\1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$ $\qquad \begin{aligned}x_1 &= y_1 - 2y_2\\ x_2 &= y_1 + y_2\end{aligned}$

$\vec{y} = B^{-1}\vec{x}$: $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $\qquad \begin{aligned}y_1 &= \tfrac{1}{3}(x_1 + 2x_2)\\ y_2 &= \tfrac{1}{3}(-x_1 + x_2)\end{aligned}$

$A = \begin{bmatrix}1&4\\2&3\end{bmatrix}$

$\lambda_1 = 5,\ \vec{b}_1 = \langle 1,1\rangle$
$\lambda_2 = -1,\ \vec{b}_2 = \langle -2,1\rangle$

(right click menu) $\longrightarrow$

$$A_B = B^{-1}AB = \frac{1}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix}\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}1&-2\\1&1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix}\begin{bmatrix}5&+2\\5&-1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}15&0\\0&-3\end{bmatrix} = \begin{bmatrix}5&0\\0&-1\end{bmatrix} \checkmark$$

$\boxed{\text{LinearAlgebra: Eigenvectors }(A) = \begin{bmatrix}-1\\5\end{bmatrix}, \begin{bmatrix}-2&1\\1&1\end{bmatrix}}$ ordering of eigenvalues still random

$\lambda = -1 \quad \lambda = 5$ columns are corresponding eigenvectors in same order

6.2b Diagonalization geometry in the plane

Okay, now its your turn. Repeat for these 3 matrices.
Use pencil so if you mess up, you dont need a new sheet.

1) $A = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}$ $\quad \vec{x}_0 = \langle 0, 5 \rangle$

2) $A = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix}$ $\quad \vec{Y}_0 = \langle 4, 2 \rangle$

3) $A = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix}$ $\quad \vec{x}_0 = \langle -4, 5 \rangle$

Breakout rooms.