

6.2 Diagonalization of square matrices

(1)

$$\frac{dx}{dt} = Ax$$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

if diagonal

$$= \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ a_{22}x_2 \\ \vdots \\ a_{nn}x_n \end{bmatrix} \Leftrightarrow \frac{d}{dt} x_i = a_{ii} x_i$$

decoupled: all t
 $x_i = c_i e$
done.

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a_{ii} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

eigenvalue \uparrow eigenvector
standard basis $\{e_i\}$ are
eigenvectors.

A general $n \times n$

Find basis $\{b_1, \dots, b_n\}$ of eigenvectors $\rightarrow B = \langle b_1, \dots, b_n \rangle$
basis changing matrix

change variables $x = By, y = B^{-1}x$

Must now re-express DEs in terms of new variables.

$$\frac{d}{dt} (By) = A(By) \rightarrow \frac{dy}{dt} = \underbrace{B^{-1}AB}_{A_B} y$$

$$B^{-1} \left[B \frac{dy}{dt} \right]$$

A_B new matrix of coefficients

6.2 Diagonalization of square matrices

(2)

But $Ab_i = \lambda_i b_i$

so $AB = A \langle b_1 | \dots | b_n \rangle$ ← each col of product equals corresponding col of right factor multiplying left factor

$$= \langle Ab_1 | \dots | Ab_n \rangle$$

$$= \langle \lambda_1 b_1 | \dots | \lambda_n b_n \rangle$$

$$= \langle b_1 | \dots | b_n \rangle \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

← each right col just scales corresponding left col

$$= B \text{diag}(\lambda_1, \dots, \lambda_n) \quad \text{shorthand!}$$

$$B^{-1} [AB = B \text{diag}(\lambda_1, \dots, \lambda_n)]$$

$$\underbrace{B^{-1}AB}_{A_B} = \text{diag}(\lambda_1, \dots, \lambda_n)$$

new coefficient matrix is diagonal!

$$\frac{dy}{dt} = A_B y \Leftrightarrow \frac{d}{dt} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{bmatrix}$$

$$\Leftrightarrow \frac{d}{dt} y_i = \lambda_i y_i \quad y_i = c_i e^{\lambda_i t}$$

$$y(0) = c_i$$

Back to x : $x = By = y_1 b_1 + \dots + y_n b_n$

$$= c_1 \underbrace{e^{\lambda_1 t} b_1}_{\text{}} + \dots + c_n \underbrace{e^{\lambda_n t} b_n}_{\text{}}$$

$\{e^{\lambda_1 t} b_1, \dots, e^{\lambda_n t} b_n\}$ basis of n -dim soln subspace

$\tau_i = \frac{1}{|\lambda_i|}$ determine timescales of exponential growth/decay along eigenvector directions
 same for all eigenvectors in a given eigenspace.

↔ straight line solutions through origin (but don't contain 0!)

$$x(0) = x_0 = By(0) = BC \rightarrow C = B^{-1}x(0) \quad \text{arb constants just new coords of } x(0).$$

6.2) Diagonalization of a square matrix

3

$$\begin{aligned} \frac{dx_1}{dt} &= -9x_1 - 4x_2 \\ \frac{dx_2}{dt} &= 6x_1 + x_2 \\ \frac{dx_3}{dt} &= -6x_1 - 4x_2 - 3x_3 \end{aligned}$$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} -9 & -4 & 0 \\ +6 & +1 & 0 \\ -6 & -4 & -3 \end{bmatrix}}^A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda = -5, -3, -3$$

$$A \rightarrow B = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 3 & 2 & 0 \\ -3 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$x = By, y = B^{-1}x$$

$$A_B = B^{-1}AB = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5y_1 \\ -3y_2 \\ -3y_3 \end{bmatrix}$$

$$\begin{aligned} y_1' &= -5y_1 & y_1 &= c_1 e^{-5t} \\ y_2' &= -3y_2 & y_2 &= c_2 e^{-3t} \\ y_3' &= -3y_3 & y_3 &= c_3 e^{-3t} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 e^{-5t} \\ c_2 e^{-3t} \\ c_3 e^{-3t} \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} +3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ -3 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9-4 \\ -9+1+1 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$\underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{y(0)}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 5e^{-5t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 4e^{-3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$\tau: \frac{1}{5}, \frac{1}{3}$
 $\sigma: 1, \frac{5}{3} \sim 2$

$$= e^{-5t} \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix} + e^{-3t} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = F \begin{bmatrix} 5e^{-5t} & -2e^{-3t} \\ -5e^{-5t} & +3e^{-3t} \\ 5e^{-5t} & -4e^{-3t} \end{bmatrix}$$