

6.1b

Eigenvectors and eigenvalues: more

①

On \mathbb{R}^n we want $\underbrace{A}_{\substack{n \times n \\ \text{matrix}}} \vec{x} = \lambda \vec{x} \rightarrow A\vec{x} = \lambda \vec{x}, \vec{x} \neq 0$
 \uparrow \uparrow
 $n \times 1$ matrix eigenvector of A
corresponding eigenvalue

Linear homogeneous system; standard form

$$\underbrace{(A - \lambda I)}_{\substack{\text{matrix}}} \vec{x} = \vec{0}$$

$|A - \lambda I| = 0$ required to have nonzero solns.

Then the coefficient matrix reduces to have at least 1 free column & the reduced system 1 free variable (there can be more!).

The soln produced by this reduction gives a basis of the soln space whose nonzero elements are eigenvectors.

The "eigenspace" for a given eigenvalue is at least 1-dimensional, but can be larger.

Review previous class 3x3 example.

then Maple tool

6.1 example 6

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \quad 0 = |A - \lambda I| = \dots = -\lambda(\lambda-1)(\lambda-3) \rightarrow \lambda = 0, 1, 3$$

(2)

$\lambda_1 = 0: A - 0I = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & -15 & -5 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_3 = t: \begin{aligned} x_1 &= 0 \\ x_1 + \frac{1}{3}x_3 &= 0 \rightarrow x_1 = -\frac{1}{3}x_3 \\ 0 &= 0 \end{aligned} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} \rightarrow b_1 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$\lambda_2 = 1: A - I = \begin{bmatrix} 3-1 & 0 & 0 \\ -4 & 6-1 & 2 \\ 16 & -15 & -5-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & -15 & -6 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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$\lambda_3 = 3: A - 3I = \begin{bmatrix} 3-3 & 0 & 0 \\ -4 & 6-3 & 2 \\ 16 & -15 & -5-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 16 & -15 & -8 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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$$B = \langle b_1, b_2, b_3 \rangle = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{2}{5} & 0 \\ \frac{1}{4} & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ 3 & 5 & 2 \end{bmatrix}$$

Maple \leftrightarrow integer choice

either choice is a basis of \mathbb{R}^3 of eigenvectors of A or an "eigenbasis."

In these simple exercises, the row reduction is easy to do by hand BUT you can use Maple to be sure of the result.

Then $x = By$, $y = B^{-1}x$ is the decoupling change of variables we need to solve $\frac{dy}{dt} = Ax$.

6.1 example 6

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6.1b Eigenvectors and eigenvalues: more

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Review:

3x3 matrix with repeated eigenvalues

$$A = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{bmatrix} \xrightarrow{\det} -\lambda^3 + 11\lambda^2 - 39\lambda + 45 = -(\lambda-5)(\lambda-3)^2 = 0 \quad (\text{factor})$$

$\lambda = 5, 3$ (solve)
 $m=1, 2 \leftarrow \text{multiplicity}$

$$\lambda=5 \quad A - 5I = \begin{bmatrix} 9-5 & 4 & 0 \\ -6 & -1-5 & 0 \\ 6 & 4 & 3-5 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row operations}}$$

$$\begin{array}{c|ccc} L & F \\ \hline 1 & 0 & -1 & \\ 0 & 1 & 1 & \\ 0 & 0 & 0 & \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = t \end{array} \quad \begin{array}{l} x_1 = t \\ x_2 = -t \\ x_3 = t \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \xrightarrow{\text{basis of 1st eigenspace}} \vec{b}_1$$

$$\lambda=3 \quad A - 3I = \begin{bmatrix} 9-3 & 4 & 0 \\ -6 & -1-3 & 0 \\ 6 & 4 & 3-3 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c|cc} L & F \\ \hline 2 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \quad \begin{array}{l} x_1 + \frac{2}{3}x_2 = 0 \\ x_2 = t_1 \\ x_3 = t_2 \end{array} \quad \begin{array}{l} x_1 = -\frac{2}{3}t_2 \\ x_2 = t_1 \\ x_3 = t_2 \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix} \quad \begin{array}{l} \xrightarrow{\text{basis of 2nd eigenspace}} \{\vec{b}_2, \vec{b}_3\} \\ \vec{b}_2 \\ \vec{b}_3 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \text{ (integers!)} \end{array}$$

$$B = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

compare Maple: $\begin{bmatrix} 1 & -2/3 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\lambda = 3$, maple does not care
 if fractions appear
 order unimportant.

change of basis:

$$\vec{x} = B \vec{y}, \vec{y} = B^{-1} \vec{x}$$

see Maple worksheet for
 2 other examples

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Eigenvectors and eigenvalues: more

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General situation on \mathbb{R}^n :

$$\underbrace{A \vec{x}}_{n \times n} = \lambda \underbrace{\vec{x}}_{n \times 1} \rightarrow (A - \lambda I) \vec{x} = 0 \quad \begin{matrix} \text{need nonzero solns} \\ \downarrow \end{matrix}$$

$0 = \det(A - \lambda I) = (-1)^n (\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0)$ [characteristic eqn for A]

$$= -(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}$$

p distinct roots λ_i of multiplicities $m_i : \sum_{i=1}^p m_i = n = \text{total # roots}$

roots are either real or complex conjugate pairs

For each real root we are guaranteed at least one free parameter in the soln of $(A - \lambda I) \vec{x} = 0$, but no more are GUARANTEED. So we can only depend on getting 1 linearly independent eigenvector for each distinct eigenvalue.

The goal is to find n linearly independent eigenvectors so that they form an ["eigenbasis"] of \mathbb{R}^n that we can use to introduce a new (linear) system of coordinates.

When eigenvalues are repeated, we can come up short and not get such an eigenbasis, so we do not get a new coord system.

If we have n distinct eigenvectors, we are guaranteed n linearly independent eigenvectors BECAUSE of the following FACT:

Eigenvectors from distinct eigenvalues are linearly independent.

6.1b1 Eigenvectors and eigenvalues: more

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distinct eigenvectors have linearly independent eigenvectors

$$1) \text{ Since if } A\vec{x} = \lambda \vec{x}, \vec{x} \neq 0 \text{ then } c(A\vec{x}) = c(\lambda \vec{x}) \\ "A(c\vec{x}) = "\lambda(c\vec{x})$$

so any multiple of an eigenvector has the same eigenvalue.

Thus two eigenvectors with distinct eigenvalues cannot be proportional, and are therefore linearly independent.

2) Suppose we have three ^{distinct} eigenvalues with $A\vec{v}_1 = \lambda_1 \vec{v}_1, A\vec{v}_2 = \lambda_2 \vec{v}_2, A\vec{v}_3 = \lambda_3 \vec{v}_3$ $\lambda_1 \neq \lambda_2 \neq \lambda_3$

Are they necessarily linearly independent? We check:

$$\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}}$$

Then $A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = A\vec{0} = \vec{0}$

$"$

$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3$

$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3 = \vec{0}$

$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_1 \vec{v}_2 + c_3 \lambda_1 \vec{v}_3 = \vec{0}$

But $\lambda_1 (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = \lambda_1 \vec{0} = \vec{0}$ subtract

$$c_1 (\underbrace{\lambda_1 - \lambda_1}_{=0} \vec{v}_1 + \underbrace{c_2 \cancel{\lambda_2 - \lambda_1}}_{\neq 0} \vec{v}_2 + \underbrace{c_3 \cancel{\lambda_3 - \lambda_1}}_{\neq 0} \vec{v}_3) = \vec{0}$$

but if both c_2 & c_3 are nonzero this implies \vec{v}_2 and \vec{v}_3 are proportional, which is not possible since they have different eigenvalues.

If either c_2 or c_3 is zero, the other one must be zero. Thus both must be zero which then implies $c_1 = 0$ from the linear relationship equation. Thus all c_1, c_2, c_3 must be zero.

Why?

This means we can put together bases of distinct eigenspaces to get a basis of the span of all the basis vectors together. If we get n linearly independent eigenvectors in \mathbb{R}^n , we get a basis of \mathbb{R}^n : an "eigenbasis"

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Eigenvectors and eigenvalues : more

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Recall how useful complex exponentials were for solns of constant coefficient single linear homogeneous DEs. The complex root pairs of the characteristic eqn for exponential rate factors led to complex conjugate exponential scalar solns.

Complex conjugate root pairs of the eigenvalue characteristic eqn lead to complex conjugate exponential vector solns of constant coefficient linear homogeneous systems of DEs.

In both cases because of linearity, the real and imaginary parts of these solns provide two independent real solutions.

Eigenvalues come in complex conjugate root pairs since they are roots of real polynomials. This extends to their corresponding eigenvectors:

$$\begin{array}{c} \vec{A}\vec{x} = \lambda \vec{x} \\ \uparrow \quad \downarrow \\ \text{real} \quad \text{complex} \end{array} \xrightarrow{\substack{\text{Take} \\ \text{Complex} \\ \text{Conjugate}}} \begin{array}{l} \overline{\vec{A}\vec{x}} = \overline{\lambda\vec{x}} \\ \overline{\vec{A}\vec{x}} = \overline{\lambda}\overline{\vec{x}} \\ = \overline{\vec{A}} \quad \overline{\vec{x}} \end{array} \quad] \text{ c.c. factors}$$

eigenvector with
complex conjugate
eigenvalue

Eigenvectors come in complex conjugate pairs

so there is no need to solve the linear system needed to obtain the eigenvector for the complex conjugate eigenvalue. We can obtain it just by taking the complex conjugate.

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Eigenvectors and eigenvalues: more

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Example: $A = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} 6-\lambda & -17 \\ 8 & -6-\lambda \end{vmatrix} = (6-\lambda)(-6-\lambda) + 136 = (\lambda-6)(\lambda+6) + 136 \\ = \lambda^2 - 36 + 136 = \lambda^2 + 100 \\ \lambda = \pm \sqrt{100} = \pm 10i$$

$\lambda = 10i$:

$$A - 10iI = \begin{bmatrix} 6-10i & -17 \\ 8 & -6-10i \end{bmatrix}$$



swap rows so
easy to make leading
coeff 1, eliminate
second row

$$\rightarrow \begin{bmatrix} 8 & -6-10i \\ 0 & 0 \end{bmatrix} \rightarrow$$

Fact If $\det A = 0$ for 2×2 matrix
then the rows are proportional.

$$0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \rightarrow ad = bc \\ \rightarrow \frac{a}{c} = \frac{b}{d}$$

In the real case it is pretty obvious,
not so in the complex case because
complex arithmetic hides the proportionality.
2nd column ratio:

$$\frac{-17}{-6-10i} = \frac{17}{(6+10i)} \frac{(6-10i)}{(6-10i)} = \frac{17(6-10i)}{36+100} \\ = \frac{17}{136}(6-10i) = \frac{6-10i}{8} \quad \checkmark \text{ first col ratio}$$

$$\begin{bmatrix} 1 & \frac{6+10i}{8} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3+5i}{4} \\ 0 & 0 \end{bmatrix}$$

Now solve the reduced equations.

$$\left[\begin{array}{cc|c} 1 & \frac{3+5i}{4} & x_1 \\ 0 & 0 & x_2 \end{array} \right] \xrightarrow{\text{L}} \left[\begin{array}{cc|c} 1 & \frac{3+5i}{4} & 0 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{F}} \left[\begin{array}{cc|c} 1 & \frac{3+5i}{4} & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow x_1 = \frac{3+5i}{4}t$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3+5i}{4}t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{3+5i}{4} \\ 1 \end{pmatrix}$$

$\begin{pmatrix} \frac{3+5i}{4} \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3+5i \\ 4 \end{pmatrix}$ is a basis
of eigenspace

b_{+} simpler for us
during toy problems

$$\underline{x = -10i}: \vec{b}_{-} = \vec{b}_{+} = \begin{pmatrix} 3-5i \\ 4 \end{pmatrix}.$$

$$\text{Done: } B = \langle b_{+} | b_{-} \rangle = \begin{bmatrix} 3+5i & 3-5i \\ 4 & 4 \end{bmatrix} \quad \text{basis changing matrix is complex.}$$