

6.10 Why this detour back to linear algebra?

(1)

Consider this coupled pair of constant coefficient linear homogeneous first order DEs:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_1 \end{aligned} \quad \left\{ \begin{array}{l} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{(scalar form)} \end{array} \right. \quad \text{or } \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \frac{dx}{dt} = Ax \quad (\text{matrix form}) \end{aligned}$$

We can't solve the first DE without knowing x_2 and can't solve the second DE without knowing x_1 . They are "coupled".

Consider their sum and difference.

$$\begin{aligned} \frac{dx_1}{dt} + \frac{dx_2}{dt} &= x_2 + x_1 \rightarrow \frac{d}{dt} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{y_1} = \underbrace{\begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}}_{y_2} \rightarrow \frac{dy_1}{dt} = y_1 \\ \frac{dx_1}{dt} - \frac{dx_2}{dt} &= x_2 - x_1 \rightarrow \frac{d}{dt} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{y_2} = \underbrace{\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}}_{y_1} \rightarrow \frac{dy_2}{dt} = -y_2 \end{aligned} \quad \left\{ \begin{array}{l} \text{decoupled!} \\ \text{exp solns!} \end{array} \right.$$

$$\text{matrix form: } \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{diagonal matrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^{-t} \end{aligned}$$

But now must go back to x_1, x_2 .

$$\begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= x_1 - x_2 \end{aligned} \quad \left\{ \begin{array}{l} \text{new variables decouple the DEs,} \\ \text{new coordinate system on } x_1-x_2 \text{ plane.} \end{array} \right.$$

Invert:

$$\begin{aligned} y_1 + y_2 &= 2x_1 \rightarrow x_1 = \frac{1}{2}(y_1 + y_2) \\ y_1 - y_2 &= 2x_2 \rightarrow x_2 = \frac{1}{2}(y_1 - y_2) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}c_1 e^t + \frac{1}{2}c_2 e^{-t} \\ &= \frac{1}{2}c_1 e^t - \frac{1}{2}c_2 e^{-t} \end{aligned} \quad \text{solved!!}$$

Matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 b_1 + y_2 b_2 = \begin{array}{c} \uparrow \quad \uparrow \\ b_1 \quad b_2 \end{array}$$

$$B = \langle b_1 | b_2 \rangle \quad \leftarrow \begin{array}{l} \text{new basis of} \\ \text{plane} \end{array}$$

$$\begin{aligned} &c_1 \underbrace{e^t}_{b_1} + c_2 \underbrace{e^{-t}}_{b_2} \\ &\{ e^t b_1, e^{-t} b_2 \} \end{aligned}$$

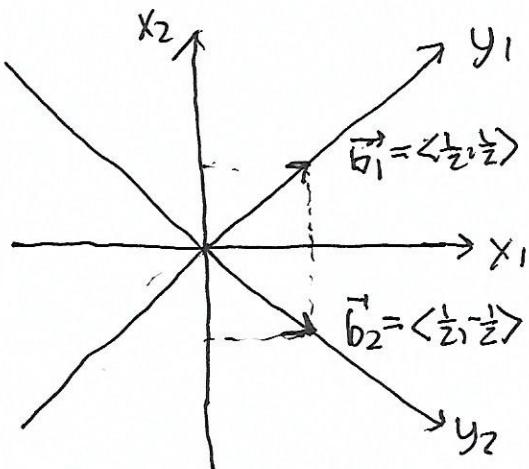
basis of 2d subspace of solns.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{B^{-1}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

6.0

Why this detour back to linear algebra?

(2)



new coord axes along \vec{b}_1, \vec{b}_2

any multiples would do the job, like
 $\langle 1, 1 \rangle, \langle 1, -1 \rangle$

It is the lines thru the origin along them which have simple straight line solns of the DEs.

Two preferred 1-d subspaces of the plane with pure exponential growth/decay.

These subspaces are determined by the coefficient matrix A :

$$Ab_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = b_1$$

$$Ab_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = -b_2$$

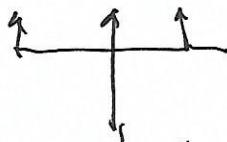
} in both cases

$$Ab_i \propto b_i$$

multiplication by A does not change line containing b_i



$$Ab_i = \lambda_i b_i$$



any nonzero vector for which

$Ax \propto x$ is called an eigenvector of A

} the constant of proportionality

$$Ax = \lambda x$$



is called the corresponding eigenvalue "lambda"

only the directions of b_1, b_2 matter.

But how to find them for any matrix A ?

They are the key to solving the DE system : $\frac{dx}{dt} = Ax$

6.1a

Eigenvalues and eigenvalues for 2×2 matrices

(1)

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2\end{aligned}$$

↑ ↑
↑ ↑
constants

scalar form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftrightarrow \frac{dx}{dt} = Ax$$

constant
coefficient matrix

symbolic form

How to find solutions to $Ax = \lambda x$?Rewrite: $Ax - \lambda x = 0$ (all unknowns on LHS)

$$\underbrace{Ax}_{2 \times 2} - \underbrace{\lambda I}_{2 \times 2} x$$

insert identity matrix into second term

$$\underbrace{(A - \lambda I)}_{2 \times 2} x = 0$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

coefficient matrix
of linear homogeneous
systemTo get nonzero solns, must not be invertible (only 0 soln!).
 $\text{rref}(A)$ must have at least one free variable column.
 $\det(A - \lambda I) = 0$ guarantees rows are proportional
have nonzero solns.

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

quadratic eqn for λ :
"characteristic equation"

Solve as precondition for finding nonzero x .For each soln λ , backsubstitute value into matrix eqn
and solve to find linearly independent solns. \rightarrow basis of
soln subspace called "eigen space"

6.1g

Eigenvectors and eigenvalues for 2×2 matrices

②

Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}$$

subtract from
main diagonal!

zero determinant:

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) \rightarrow \lambda = 1, -1$$

eigenvalues

$\lambda = 1$:

$$(A - I)x = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1 & x_1 \\ 0 & 0 & x_2 \end{array} \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_2 = 0 \\ 0 = 0 \end{cases}$$

$\begin{matrix} x_1 & x_2 \\ L & F \end{matrix}$

$$\begin{aligned} x_2 &= t & \rightarrow x_1 &= t \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

choose b_1 (simplest)

[basis vector for eigenspace $\lambda = 1$]

$\lambda = -1$:

$$(A + I)x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 0 & x_2 \end{array} \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x_1 + x_2 = 0 \\ 0 = 0 \end{cases}$$

$\begin{matrix} x_1 & x_2 \\ L & F \end{matrix}$

$$\begin{aligned} x_2 &= t & \rightarrow x_1 &= -t \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

b_2

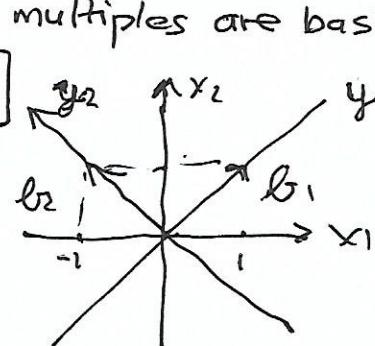
[basis vector for eigenspace $\lambda = -1$]

Note $b_1 = \frac{1}{2}b_2$, $b_2 = -\frac{1}{2}b_1$ any nonzero multiples are basis vectors.

Basis changing matrix $B = \langle b_1 | b_2 \rangle = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

same axis lines,
different tickmarks

we are free to choose
convenient multiples for
basis eigenvectors



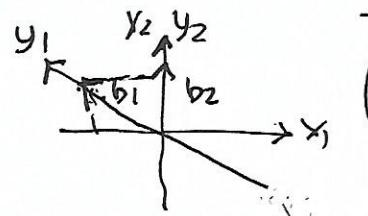
6.19

Eigenvectors and eigenvalues for 2×2 matrices

(3)

In class student exercises:

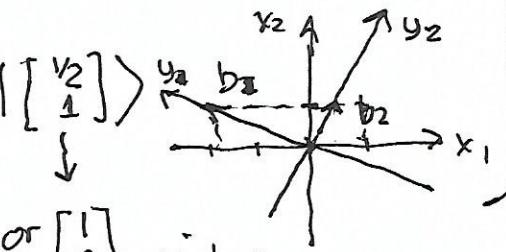
a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \rightarrow \lambda = 2, 3$
 $B = \left\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mid \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$



draw
eigenvectors
and new
axes

b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \lambda = 0, 5$
 $B = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle$

or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ simpler
if want integer eigenvectors.



see Maple worksheet EigenExploreDrag2.mw
for graphical exploration looking for directions where
 Ax and x line up, meaning $Ax \propto x$.