

5.6b Driven damped harmonic oscillators: Special cases

(1)

$$y'' + k_0 y' + \omega_0^2 y = B_0 \cos \omega t \quad \text{we assumed } k_0 > 0$$

\downarrow
 $\omega_0 = 0$? what changes? Two cases: 1) $\omega \neq \omega_0$ no problem formulas still valid

No damping

2) $\omega = \omega_0$ undamped resonance.

Undamped resonance

Easy example

$$y'' + y = \cos t$$

$$(D^2 + 1)y = \cos t$$

$$\begin{aligned} r &= \pm i \\ m &= 1 \end{aligned}$$

$$\begin{aligned} r &= \pm i \\ m &= 1 \end{aligned} \quad (D^2 + 1) \cos t = 0$$

$$(D^2 + 1)^2 y = (D^2 + 1) \cos t = 0$$

$$\begin{aligned} r &= \pm i \\ m &= 2 \end{aligned} \quad \rightarrow y = (C_1 + C_3 t) \cos t + (C_2 + C_4 t) \sin t$$

$$= (C_1 \cos t + C_2 \sin t) + t(C_3 \cos t + C_4 \sin t)$$

y_p fixed amplitude
"natural behavior"
set by initial conditions

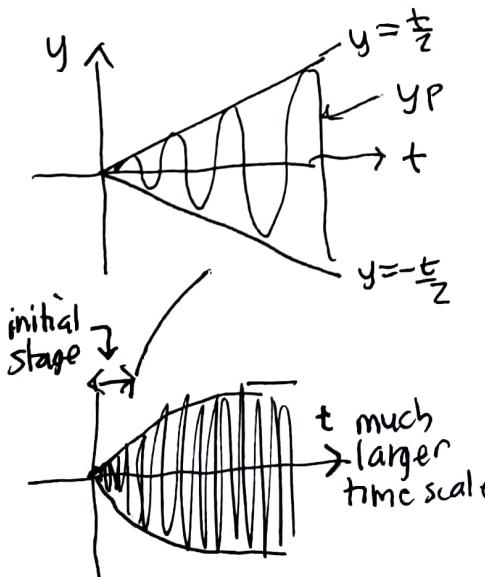
y_p linearly growing amplitude
response soln

$$1 \quad \sum y_p = t(C_3 \cos t + C_4 \sin t)$$

$$0 \quad [y_p] = t(-C_3 \sin t + C_4 \cos t) + (C_3 \cos t + C_4 \sin t)$$

$$1 \quad [y_p] = t(-C_3 \cos t - C_4 \sin t) + (-C_3 \sin t + C_4 \cos t) + (-C_3 \sin t + C_4 \cos t)$$

$$\underline{y_p'' + y_p = t(C_3 - C_3) \cos t + (C_4 - C_4) \sin t} \quad \begin{aligned} -2C_3 \sin t + 2C_4 \cos t &= \cos t \\ \underline{= 0} & \underline{= 1} \\ C_3 = 0, C_4 = \frac{1}{2} & \quad y_p = \frac{t}{2} \sin t \end{aligned}$$



no matter what C_1, C_2 are eventually the response function dominates and grows without bound

UNLESS the system breaks or very small damping is present, and it eventually reaches a maximum amplitude

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Undamped resonance: general case : $\omega = \omega_0$, $\nu_0 = 0$

$$(D^2 + \omega_0^2)y = y'' + \omega_0^2 y = B_0 \cos \omega_0 t$$

$$\text{same roots} \rightarrow (D^2 + \omega^2)^2 y = 0$$

$$\left. \begin{array}{l} r = \pm i\omega \\ m=2 \end{array} \right\} y = (c_1 + c_3 t) \cos \omega t + (c_2 + c_4 t) \sin \omega t$$

$$= \underbrace{(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t)}_{\text{H}_1} + \underbrace{t(c_3 \cos \omega_0 t + c_4 \sin \omega_0 t)}_{\text{H}_2}$$

fixed amplitude

\rightarrow
growing amplitude

$$u_0^2 [y_p = + (c_3 \cos \omega t + c_4 \sin \omega t)$$

$$\partial [y_p' = t(-c_3 \omega_0 \sin \omega_0 t + c_4 \omega_0 \cos \omega_0 t) + (c_3 \cos \omega_0 t + c_4 \sin \omega_0 t)$$

$$y_p'' = t(-c_3\omega_0^2 \cos \omega_0 t - c_4\omega_0^2 \sin \omega_0 t) + (-c_3\omega_0 \sin \omega_0 t + c_4\omega_0 \cos \omega_0 t) \\ + (-c_3\omega_0 \sin \omega_0 t + c_4\omega_0 \cos \omega_0 t)$$

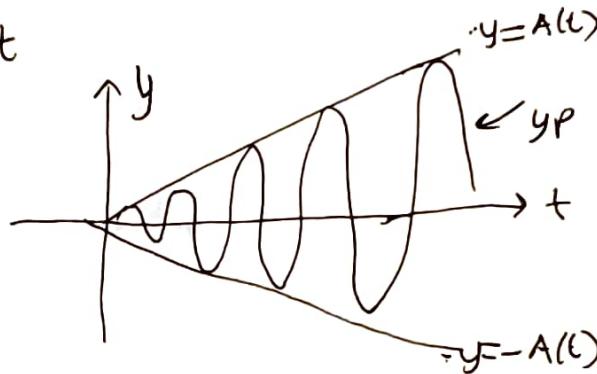
$$y_p'' + \omega_0^2 y_p = t \left(C_3 \omega_0^2 \cos \omega_0 t + (C_4 \omega_0^2 \sin \omega_0 t) \right) + \underbrace{-2C_3 \omega_0 \sin \omega_0 t}_{=0} + \underbrace{2C_4 \omega_0 \cos \omega_0 t}_{=1}$$

$$= B_0 \cos \omega_0 t$$

$$C_3 = 0$$

$$C_4 = \frac{B_0}{2\omega_0}$$

$$y_p = \underbrace{\frac{B_0}{2\omega_0}}_{A(t)} t \sin \omega_0 t$$



the sine is 90 degrees behind driving cosine,
as occurs in resonance

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Damped Driven Harmonic oscillators: Special cases

(3)

no damping but $\omega \neq \omega_0$ set $b_0 = 0$:

$$y_p = \frac{B_0}{(\omega_0^2 - \omega^2)^2 + D_0^2 \omega^2} [B\omega_0^2 - \omega^2] \cos \omega t + \frac{B\omega_0 D_0 \omega}{(\omega_0^2 - \omega^2)^2 + D_0^2 \omega^2} \sin \omega t \rightarrow 0$$

$$= \frac{B_0}{(\omega_0^2 - \omega^2)^2} [(\omega_0^2 - \omega^2) \cos \omega t] = \underbrace{\frac{B_0}{\omega_0^2 - \omega^2}}_{\text{big response when } \omega \approx \omega_0} \cos \omega t \leftarrow \text{in phase with driving function } f(t) = B_0 \cos \omega t$$

BEATING impose initial conditions $y(0) = 0, y'(0) = 0$
at equilibrium, at rest

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{B_0}{\omega_0^2 - \omega^2} \cos \omega t$$

$$y' = -c_1 \omega_0 \sin \omega_0 t + c_2 \omega_0 \cos \omega_0 t - \frac{B_0 \omega}{\omega_0^2 - \omega^2} \sin \omega t$$

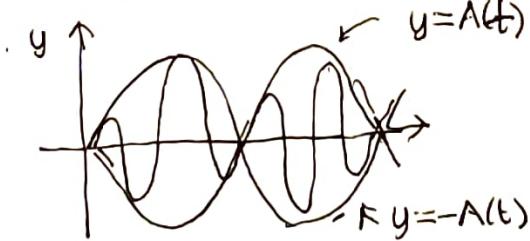
$$y(0) = c_1 + \frac{B_0}{\omega_0^2 - \omega^2} = 0 \quad c_1 = -\frac{B_0}{\omega_0^2 - \omega^2}$$

$$y'(0) = c_2 \omega_0 = 0 \rightarrow c_2 = 0$$

$$y = \frac{B_0}{\omega_0^2 - \omega^2} (\underbrace{\cos \omega t - \cos \omega_0 t}_{-2 \sin \frac{\omega - \omega_0}{2} t + \sin \frac{(\omega + \omega_0)}{2} t} + 2 \sin \frac{(\omega + \omega_0)}{2} t \sin \frac{(\omega_0 - \omega)}{2} t)$$

$$= \frac{2B_0}{\omega_0^2 - \omega^2} \underbrace{\sin \frac{(\omega - \omega_0)}{2} t}_{\pm \omega_-} \underbrace{\sin \frac{(\omega + \omega_0)}{2} t}_{\omega_+}$$

$A(t)$
longer period oscillation
"modulates" shorter period oscillation



subtraction identity:

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

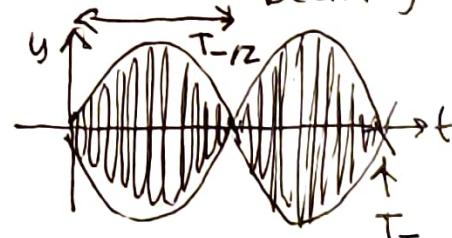
$$\omega_+ = \frac{1}{2}(\omega_0 + \omega) > \omega = \frac{1}{2}(\omega_0 - \omega_-)$$

$$T_+ = \frac{2\pi}{\omega_+} < T_- = \frac{2\pi}{\omega_-}$$

shorter longer
(beat frequency)

if $\omega \approx \omega_0$, then $T_- \gg T_+ \approx T_0$

$\omega_+ \approx \omega_0$ "Beating"



High school physics: tuning forks of nearby frequencies — hear dramatic beating of intensity.