

5.5a

Nonhomogeneous constant coefficient linear DEs : method of undetermined coefficients

①

Standard form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f(x)$$

$$(D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0)y = f(x)$$

P(D) polynomial in D

↑ now RHS not zero
what to do?

homogeneous

Sln:

$$y = e^{rx} \rightarrow \text{DE}$$

$$P(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$$

$$= (r - r_1)^{m_1} \dots$$

↓

$$y = y_h + y_p$$

roots, multiplicities determine general soln: $y = y_h \rightarrow$

↓

$$P(D)(y) = f(x)$$

$$P(D)(y_h + y_p) = f(x)$$

$$\begin{aligned} P(D)(y_h) &+ P(y_p) = f(x) \\ \approx 0 & \qquad \qquad \qquad P(D)(y_p) = f(x) \end{aligned}$$

We just need to find one "particular" soln of the nonhomogeneous DE and add it to the general soln of the related homogeneous DE to get the general soln.

First order case: $e^{px}[y' + py = f(x)] \rightarrow \frac{d}{dx}(ye^{px}) = f(x)e^{px}$

$$\downarrow \int e^{px} dx = e^{px}$$

$$ye^{px} = \underbrace{\int f(x)e^{px} dx}_{} + C$$

need exact antiderivative to proceed,

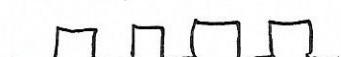
$$\text{then } y = \underbrace{e^{-px} \int f(x)e^{px} dx}_{} + \underbrace{Ce^{-px}}_{y_h}$$

An integral formula exists for the particular soln but requires functions which can be integrated against an exponential.

The "variation of parameters" method generalizes this but we don't need it here.

Instead the "method of undetermined coefficients" covers most applications to physical problems.

→ Chapter 10, Laplace transforms is similar — integral formulas for solns. Useful for piecewise RHS functions!



etc

studied in
Engineering:
(EE!)

S.Sq

Method of undetermined coefficients

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This second technique "reverse engineers" our soln technique for the homogeneous case to make it work to solve the nonhomogeneous case for RItS functions $f(x)$ which are linear combinations of solns of some homogeneous constant coefficient linear DE:

$f(x) \sim$ exps, sines, cosines, polynomials, products of these
 Concretely: $P(D)y_i = f_i(x) \rightarrow P(D)\underbrace{\left(\sum_{i=1}^P y_i\right)}_{\text{combine together}} = \sum_{i=1}^P f_i(x)$
 $\quad \quad \quad = \sum_{i=1}^P P(D)y_i \uparrow$
 by linearity

We look at various $f_i(x)$ choices, the results can then be added together.

reverse engineering ? 3 examples:

(1) $f(x) = 2xe^{-3x}$ is a special case of $\underbrace{(c_1 + c_2x)e^{-3x}}_{m=2}$ $\downarrow r = -3$ gensoln for hom DE with $r = -3, m = 2$

$$(r+3)^2 = 0 \rightarrow (D+3)^2 y_h = 0 \rightarrow (D^2 + 6D + 9) y_h = 0$$

From the "root structure" of the RItS, we find the DE of which it is a soln.
 $\text{so } (D+3)^2 f(x) = 0$

(2) $f(x) = 2 \cos 3x$ is a special case of $\underbrace{c_1 \cos 3x + c_2 \sin 3x}_{m=1}$ $\downarrow r = \pm 3i$ gensoln for hom DE with $r = \pm 3i, m = 1$

$$0 = (r-3i)(r+3i) = r^2 + 9 \rightarrow (D^2 + 9) y_h = 0 \quad \text{so } (D^2 + 9) f(x) = 0$$

(3) $f(x) = 3x \sin 2x$ is a special case of $\underbrace{(c_1 + c_2x) \cos 2x +}_{m=2} \underbrace{(c_3 + c_4x) \sin 2x}_{r = \pm 2i}$

$$0 = (r-2i)^2(r+2i)^2 = (r^2 + 4)^2 \rightarrow (D^2 + 4)^2 y_h = 0 \quad \text{so } (D^2 + 4)^2 f(x) = 0.$$

In each case we worked backwards to find the differential operator which when applied to $f(x)$ gives zero.

The next example shows how we can use this idea.

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Method of Undetermined Constants

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$$\begin{aligned}
 P(D)y &= (\underbrace{D+1}_{(D^2+3D+2)})(D+2)y = 2x e^{-3x} \\
 &= (D^2+3D+2)y \\
 &= y'' + 3y' + 2y
 \end{aligned}$$

RHS root structure
 $r= -3, m=2$
 $r= -1, m=1$
 $r= -2, m=1$

$(r+1)(r+2) = 0$
 LHS root structure
 no "overlapping" roots

$$\begin{aligned}
 (D+3)^2(D+1)(D+2)y &= (D+3)^2(2x e^{-3x}) = 0
 \end{aligned}$$

hom case! but
 higher order
 arb. constants in solns

$r= -3, m=2$
 $r= -2, m=1$
 $r= -1, m=1$

$$\begin{aligned}
 (r+3)^2(r+1)(r+2) &= 0 \rightarrow y = \underbrace{c_1 e^{-x}}_{y_n} + \underbrace{c_2 e^{-2x}}_{y_n} + \underbrace{(c_3 + c_4 x) e^{-3x}}_{y_p}
 \end{aligned}$$

extra terms

Backsub into original DE to determine extra constants:

$$\cdot P(D)((c_3 + c_4 x) e^{-3x}) = 2x e^{-3x}$$

$\frac{1}{1}$
 undetermined coeffs in LHS

concrete coeffs on RHS
 gives linear system of eqns for c_3, c_4

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Method of undetermined coefficients

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$$y_p = (c_3 + c_4 x) e^{-3x} \rightarrow y_p'' + 3y_p' + 2y_p = 2x e^{-3x}$$

Details are tedious. Proceed with caution.

$$y_p = (c_3 + c_4 x) e^{-3x}$$

$$y_p' = [c_4 - 3(c_3 + c_4 x)] e^{-3x} = [(-3c_3 + c_4) - 3c_4 x] e^{-3x}$$

$$y_p'' = [-3c_4 - 3[(-3c_3 + c_4) - 3c_4 x]] e^{-3x}$$

$$= [(9c_3 - 6c_4) + 9c_4 x] e^{-3x}$$

$$2y_p = (2c_3 + 2c_4 x) e^{-3x}$$

$$3y_p' = [(-9c_3 + 3c_4) - 9c_4 x] e^{-3x}$$

$$y_p'' = [(9c_3 - 6c_4) + 9c_4 x] e^{-3x}$$

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= [(2-9+9)c_3 + (3-6)c_4 + (2-9+9)c_4 x] e^{-3x} \\ &= [\underbrace{(2c_3 - 3c_4)}_{=0} + \underbrace{2c_4 x}_{=2}] e^{-3x} = \underbrace{2x e^{-3x}}_{=2} + \underbrace{0 e^{-3x}}_{=0} \end{aligned}$$

coefficients of linearly ind. functions are unique

e^{-3x}, xe^{-3x}
must be same on LHS & RHS

$$2c_3 - 3c_4 = 0 \rightarrow c_3 = \frac{3}{2}c_4 = 3/2$$

$$2c_4 = 2 \rightarrow c_4 = 1 \quad \uparrow$$

$$y_p = \left(\frac{3}{2} + x\right) e^{-3x}$$

so $\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \left(\frac{3}{2} + x\right) e^{-3x}}$ ← Impose I.C.S!
 $y(0) = 0, y'(0) = 0$

IVP: $y' = -c_1 e^{-x} - 2c_2 e^{-2x} + \underbrace{[1 - 3(\frac{3}{2} + x)] e^{-3x}}_{-7/2 - 3x}$

$$y(0) = c_1 + c_2 + 3/2 = 0$$

$$y'(0) = -c_1 - 2c_2 - 7/2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2+1} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \end{bmatrix} \cdot \frac{1}{2} = -\frac{1}{2} \begin{bmatrix} 6 & -7 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}$$

$$\boxed{y = \frac{1}{2} e^{-x} - 2e^{-2x} + \left(\frac{3}{2} + x\right) e^{-3x}}$$

5.5a Method of undetermined coefficients

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Now consider "overlapping" roots between LHS & RHS. (same LHS!)

Change RHS to $\frac{3}{2}x e^{-2x}$

$$(D+1)(D+2)y = \frac{3}{2}x e^{-2x}$$

$$(r+1)(r+2) = 0$$

$$r=-1 \quad r=-2$$

$$m=1 \quad m=1$$

overlapping roots

$$\left. \begin{array}{l} r=-2 \\ m=2 \end{array} \right\} \rightarrow \begin{array}{l} (r+2)^2 = 0 \\ (D+2)^2 (\frac{3}{2}x e^{-2x}) = 0 \end{array}$$

$$(D+2)^2 (D+1)(D+2)y = (D+2)^2 (\frac{3}{2}x e^{-2x}) = 0$$

$$\left. \begin{array}{l} r=-2 \\ m=1 \end{array} \right\} \rightarrow \begin{array}{l} r=-1 \\ m=1 \end{array}$$

$m=3$
quadratic
coeff.

$$y = c_1 e^{-x} + (c_2 + c_3 x + c_4 x^2) e^{-2x}$$

$$= \underbrace{c_1 e^{-x}}_{y_h} + \underbrace{c_2 e^{-2x}}_{\dots} + \underbrace{(c_3 x + 4x^2)}_{y_p} e^{-2x}$$

extra terms

Backsub y_p into

$$y'' + 3y' + 2y = \frac{3}{2}x e^{-2x}$$

to determine c_3, c_4 .

Get general soln.

Maple finds $y_p = (-\frac{3}{2}x - \frac{3}{4}x^2) e^{-2x}$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + (-\frac{3}{2}x - \frac{3}{4}x^2) e^{-2x}} \quad \text{gen soln.}$$

Next can impose initial conditions.

(see Maple)

Then another similar pair of examples.

driven constant coefficient linear DEs (method of undetermined coefficients)

$$\underbrace{y'' + 4y' + 4y = e^x + e^{-x}}_{(D^2 + 4D + 4)y}, \quad y(0) = 2, \quad y'(0) = 0$$

$$\underbrace{(D^2 + 4D + 4)y}_{r^2 + 4r + 4 = 0}$$

$$(r+2)^2 = 0$$

$$r = -2 \quad \text{mult: 2}$$

$$y_h = (C_1 + C_2x)e^{-2x}$$

homogeneous soln.

$y = y_h + y_p$ need to find
particular soln y_p

$f(x)$ is itself a soln of a const coeff linear hom DE:

$$\underbrace{e^x + e^{-x}}_{r=1 \quad r=-1} \rightarrow (r-1)(r+1) = 0$$

$$(D-1)(D+1)f(x) = 0$$

$$(D-1)(D+1)[(D+2)^2 y = e^x + e^{-x}]$$

$$(D-1)(D+1)(D+2)^2 y = (D-1)(D+1)(e^x + e^{-x}) = 0$$

4th order hom DE.

$$(r-1)(r+1)(r+2)^2 = 0$$

$$r = -2, 1, -1$$

$$\text{mult: } 2, 1, 1$$

$$4\text{th order DE gensln: } y = \underbrace{(C_1 + C_2x)e^{-2x}}_{y_h} + \underbrace{C_3 e^x + C_4 e^{-x}}_{y_p}$$

not a soln of 2nd order D.E., backsub y_p to fix constants
to make it a soln.

$$1 (y_p = C_3 e^x + C_4 e^{-x})$$

$$4 (y_p' = C_3 e^x - C_4 e^{-x})$$

$$1 (y_p'' = C_3 e^x + C_4 e^{-x})$$

$$y_p'' + 4y_p' + 4y_p = (4+4+1)C_3 e^x + (4-4+1)C_4 e^{-x} \\ = \underbrace{9C_3 e^x}_{1} + \underbrace{C_4 e^{-x}}_{1} = e^x + e^{-x} \rightarrow C_3 = \frac{1}{9}, C_4 = 1 \rightarrow y_p = \frac{1}{9}e^x + e^{-x}$$

$$y = (C_1 + C_2x)e^{-2x} + \frac{1}{9}e^x + e^{-x}$$

gensln of 2nd order DE.

solve ICS

$$y = (C_1 + C_2x)e^{-2x} + \frac{1}{9}e^x + e^{-x}$$

$$y' = [C_2 + (-2)(C_1 + C_2x)]e^{-2x} + \frac{1}{9}e^x - e^{-x}$$

$$y(0) = C_1 + \frac{1}{9} + 1 = 2 \rightarrow C_1 = \frac{8}{9}$$

$$y'(0) = -2C_1 + C_2 + \frac{1}{9} - 1 = 0 \rightarrow C_2 = \frac{8}{9} + 2(\frac{8}{9}) \\ = 3(\frac{8}{9})$$

$$y = \frac{8}{9}(1+3x)e^{-2x} + \frac{1}{9}e^x + e^{-x} \quad \text{IVP soln}$$

complication: some new roots coincide with some old roots

$$\underbrace{y'' + 4y' + 4y = e^{2x} + e^{-2x}}_{(D+2)^2 y \text{ same as before}} \quad y(0) = 2, \quad y'(0) = 0$$

$$(D-2)(D+2)(D+2)^2 y = 0 \quad (D-2)(D+2)(e^{2x} + e^{-2x}) = 0$$

$$(r-2)(r+2)^3 = 0$$

$$r=2 \quad r=-2 \quad \text{mult: 3}$$

$$y = (C_1 + C_2x + C_3x^2)e^{-2x} + C_4e^{2x}$$

$$y_p = C_3x^2e^{-2x} + C_4e^{2x}$$

HW: now backsub y_p into original DE to determine C_3, C_4 ; then solve ICS.

* CHECK WITH dsolve

Note: e^{-2x} satisfies hom. DE

* $x \rightarrow xe^{-2x}$ satisfies hom. DE

* $x \rightarrow x^2e^{-2x}$ does not, ok

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Method of undetermined coefficients

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Bonus example (easier)

$$(D+1)(D+2)y$$

$$= y'' + 3y' + 2y = 1 + 6e^{-3x}$$

$$\begin{array}{l} (r+1)(r+2) = 0 \\ r=-1 \quad r=-2 \\ m=1 \end{array} \quad \begin{array}{l} e^{0x} = 1 \\ r=0 \\ m=1 \end{array} \quad \left. \begin{array}{l} r=-3 \\ m=1 \end{array} \right\} \quad \begin{array}{l} (r-0)(r+3) = r(r+3) = 0 \\ D(D+3)(1+6e^{-3x}) = 0 \end{array}$$

$$D(D+3) (D+1)(D+2)y = D(D+3) (1 + 6e^{-3x}) = 0$$

$$y = \underbrace{c_1 e^{-x} + c_2 e^{-2x}}_{y_h} + \underbrace{c_3 + c_4 e^{-3x}}_{y_p}$$

$$\begin{array}{ccc} 1 + 6e^{-3x} & & \\ \downarrow & \downarrow & \\ c_3 & c_4 e^{-3x} & \end{array}$$

when no root overlap,
just replace each term by
corresponding gen soln of which
it is a special case.

Back sub:

$$2 [y_p = c_3 + c_4 e^{-3x}]$$

$$3 [y_p' = 0 - 3c_4 e^{-3x}]$$

$$1 [y_p'' = 9c_4 e^{-3x}]$$

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= \underbrace{2c_3}_{=1} + \underbrace{(2-9+9)c_4 e^{-3x}}_{=0} = 1 + 6e^{-3x} \\ c_3 &= \frac{1}{2} \\ c_4 &= 3 \rightarrow y_p = \frac{1}{2} + 3e^{-3x} \end{aligned}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2} + 3e^{-3x}$$

5.5a Nonhomogeneous C.C.L.DES: method of undetermined coeffs (8)

Summary bypassing solving higher order hom. DE]

If any term in the trial particular soln (= gen. soln. for a given type of R.H.S term) satisfies the hom. DE,
multiply it by x.

If any term in the new trial function satisfies the hom. DE,
multiply it by x.

Stop when no term satisfies the hom. DE.

Example

$$(D^3 + D^2)y = D^2(D+1)y = 3e^x + 4x^2$$

$$r^2(r+1) = 0 \leftarrow$$

$$\begin{array}{ll} r=0 & r=-1 \\ m=2 & m=0 \end{array}$$

$$Y_h = C_1 e^{-x} + (C_2 + C_3 x) e^{0x} \stackrel{0x}{=} 1$$

$$\begin{aligned} Y_{(1)p} &= C_4 e^x \\ Y_{(1)p}' &= C_4 e^x \end{aligned}$$

$$1 [Y_{(1)p}'' = C_4 e^x]$$

$$1 [Y_{(1)p}''' = C_4 e^x]$$

$$Y_{(1)p}''' + Y_{(1)p}'' = 2C_4 e^x = 3e^x$$

$$2C_4 = 3 \rightarrow C_4 = 3/2$$

$$Y_{(1)p} = \frac{3}{2} e^x \checkmark$$

$$Y_p = Y_{(1)p} + Y_{(2)p}$$

$$= \frac{3}{2} e^x + 4x^2 - 4/3 x^3 + 1/3 x^4 \checkmark$$

$$Y = Y_h + Y_p \text{ etc.}$$

$$\left. \begin{array}{l} \downarrow \\ Y_{(2)p} = \underbrace{C_5 + C_6 x + C_7 x^2}_{\text{HOM}} \\ \cancel{x} \\ Y_{(2)p} = \underbrace{C_5 x + C_6 x^2 + C_7 x^3}_{\text{HOM}} \\ \cancel{x} \\ Y_{(2)p} = C_5 x^2 + C_6 x^3 + C_7 x^4 \\ Y_{(2)p}' = 2C_5 x + 3C_6 x^2 + 4C_7 x^3 \end{array} \right\} \text{NOPE, OK}$$

$$\begin{aligned} 1 [Y_{(2)p}''' &= 2C_5 + 6C_6 x + 12C_7 x^2] \\ 1 [Y_{(2)p}'' &= 6C_6 + 24C_7 x] \end{aligned}$$

$$Y_{(2)p}''' + Y_{(2)p}'' = \underbrace{(2C_5 + 6C_6)}_0 + \underbrace{(6C_6 + 24C_7 x)}_0 + \underbrace{12C_7 x^2}_4$$

$$12C_7 = 4 \rightarrow C_7 = 1/3 \downarrow$$

$$6C_6 + 24C_7 = 0 \rightarrow C_6 = -4C_7 = -4/3 \checkmark$$

$$2C_5 + 6C_6 = 0 \rightarrow C_5 = -3C_6 = 4$$

$$Y_{(2)p} = 4x^2 - 4/3 x^3 + 1/3 x^4$$