

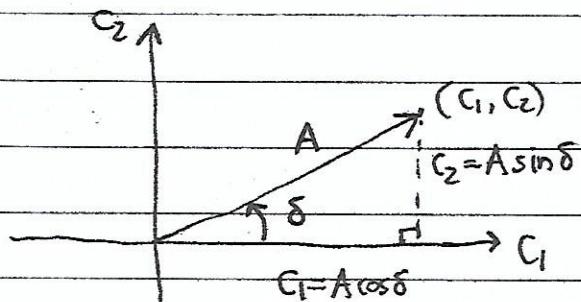
Trig identities and relative phase of sinusoidal functions

1

Any linear combination of cosines and sines of the same frequency $\omega > 0$ can be rewritten as a "phase-shifted cosine" using the cosine addition identity:

Therefore $C_1 = A \cos\delta$ is the relationship between the
 $C_2 = A \sin\delta$

two different parametrizations of the same "sinusoidal functions of frequency ω ". This is the exact same 2-parameter family of

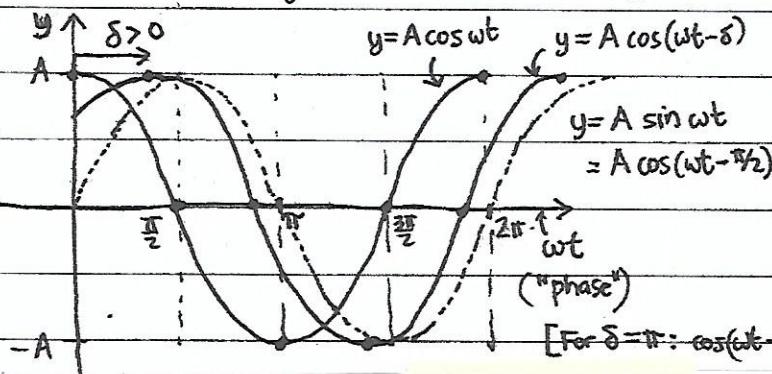


relationship which exists between Cartesian coordinates (x, y) in the x - y plane and polar coordinates (r, θ) . Thus one finds the amplitude and phase shift of a sinusoidal function given in the linear form $[C_1, C_2]$ specified by finding the equivalent polar coordinates (A, δ) .

of the point (C_1, C_2) in the "linear parameter space", each point of which represents a sinusoidal function. The inverse relationship is

$$A = \sqrt{(c_1)^2 + (c_2)^2} \quad , \quad \tan \delta = \frac{c_2}{c_1}$$

The phase shift δ is usually chosen uniquely from the interval $-\pi < \delta \leq \pi$.
 [See next sheet for how to do this using the inverse tangent.]



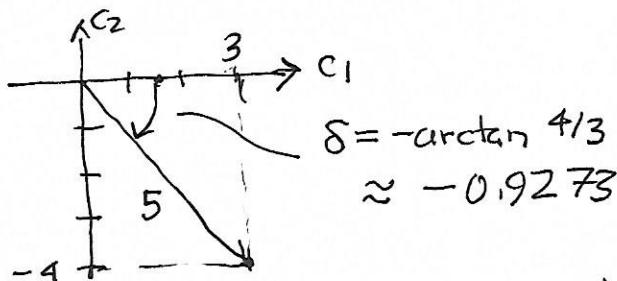
For $0 < \delta < \pi$, the phase-shifted cosine "lags" the cosine (its peaks occur after the cosine peaks – or at larger t values).

For $-180^\circ < \delta < 0$, the phase-shifted cosine "leads" the cosine (its peaks occur before the cosine peaks— or at smaller t values).
 $= -\cos(\omega t)$, said to be 180° out of phase.]

Sinusoidal functions : choosing the phase shift angle

(3)

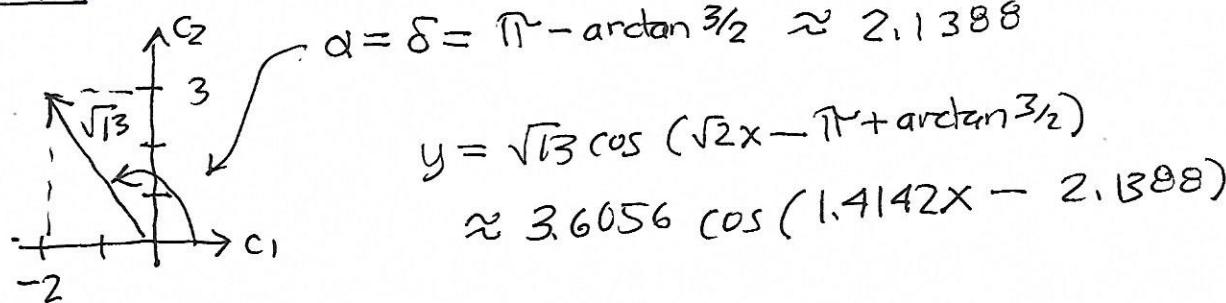
Example $y = 3 \cos 2x - 4 \sin 2x \rightarrow A = \sqrt{3^2 + 4^2} = 5$



$$\alpha = 2\pi - \arctan \frac{4}{3} \\ \approx 5.3559$$

$$y = 5 \cos(2x + \arctan \frac{4}{3}) = 5 \cos(2x - 2\pi + \arctan \frac{4}{3}) \\ \approx 5 \cos(2x + 0.9273) \approx 5 \cos(2x - 5.3559)$$

Example $y = -2 \cos \sqrt{2}x + 3 \sin \sqrt{2}x \rightarrow A = \sqrt{2^2 + 3^2} = \sqrt{13}$



$$\alpha = \delta = \pi - \arctan \frac{3}{2} \approx 2.1388$$

$$y = \sqrt{13} \cos(\sqrt{2}x - \pi + \arctan \frac{3}{2}) \\ \approx 3.6056 \cos(1.4142x - 2.1388)$$

Exercise $y = -15 \cos 3x - 10 \sin 3x$ (multiples of 5)

1) plot $(c_1, c_2) = (-15, -10) = (-3.5, -2.5)$

2) evaluate A

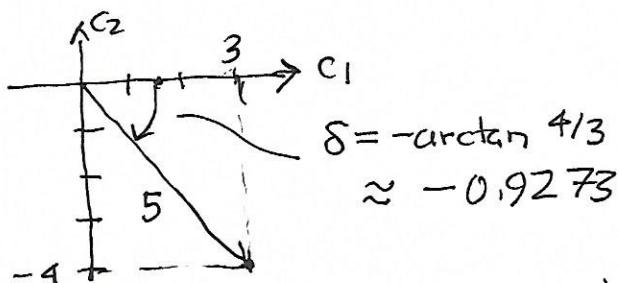
3) evaluate α and δ exactly, then approximately.

4) Rewrite y in phase-shifted form with both angles as above.

Sinusoidal functions : choosing the phase shift angle

(3)

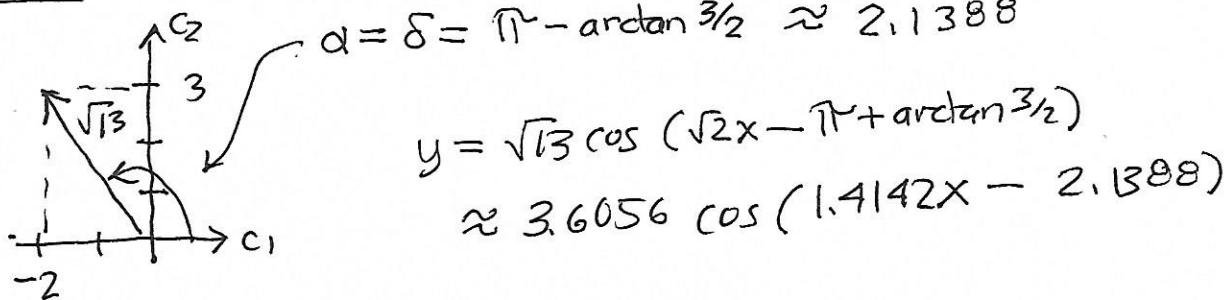
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Exponentially decaying sinusoidal functions

(4)

2nd order linear homogeneous constant coefficient DEs have solutions which are products of a real exponential and a sinusoidal factor when the roots of the characteristic equation are complex.

Example

$$y'' + 6y' + 13y = 0 \rightarrow r^2 + 6r + 13 = 0 \rightarrow r = -3 \pm 2i$$

$$y(0) = 2, y'(0) = -4$$

$$e^{rx} = e^{(-3 \pm 2i)x} = e^{-3x} (\cos 2x \pm i \sin 2x)$$

$$\downarrow e^{-3x} \cos 2x, e^{-3x} \sin 2x$$

$$\downarrow y = e^{-3x} (c_1 \cos 2x + c_2 \sin 2x)$$

impose initial conditions:

$$c_1 = 2, c_2 = 1$$

$$y = e^{-3x} \underbrace{(2 \cos 2x + \sin 2x)}_{\sqrt{5} \cos(2x - \arctan \frac{1}{2})} = \underbrace{(\sqrt{5} e^{-3x})}_{A(x)} \cos(2x - \arctan \frac{1}{2})$$

A(x) decaying amplitude

$$y = \pm A(x) \text{ when } \cos(2x - \arctan \frac{1}{2}) = \pm 1$$

(max, min values)

$$\tilde{\nu} = \frac{1}{3}, 5\tilde{\nu} = \frac{5}{3} \approx 1.67$$

$$\text{period } T = \frac{2\pi}{2} = \pi \approx 3.14$$

$$\frac{T}{5\tilde{\nu}} \approx 1.88 \quad \text{or} \quad \frac{5\tilde{\nu}}{T} \approx 0.53 \quad \text{about } \frac{1}{2} \text{ period}$$

in one "decay window"

plot $\sqrt{5}e^{-3x}$, $-\sqrt{5}e^{-3x}$, $e^{-3x}(2\cos 2x + \sin 2x)$ together

(see Maple)

envelope functions
oscillates between envelope functions

$$\underline{\text{Exercise}} \quad 4y'' + 4y' + 17y = 0, y(0) = -12, y'(0) = -10$$

$$\text{Maple} \hookrightarrow y = e^{-x/2} \underbrace{(-12 \cos 2x - 8 \sin 2x)}$$

\leftarrow rewrite, calculate A_0, δ, α

$$= \underbrace{A_0 e^{-x/2}}_{\leftarrow} \cos(2x - \delta)$$

$\hookrightarrow y = \pm A_0 e^{-x/2}$ envelope functions

plot $A_0 e^{-x/2}, -A_0 e^{-x/2}, e^{-x/2} \underbrace{(-12 \cos 2x - 8 \sin 2x)}$ together using online Maple template
 \leftarrow should touch these at extremes of