

5.3a

Higher order linear homogeneous DEs and complex exponentials

What to do with complex exponentials?

Let's work thru an explicit example.

(1)

$$y'' + 6y' + 13y = 0 \xrightarrow{y = e^{rx}} (r^2 + 6r + 13) e^{rx} = 0$$

"characteristic eqn": $r^2 + 6r + 13 = 0$

$$r = \frac{-6 \pm \sqrt{36 - 4(13)}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

$$e^{rx} = e^{(-3 \pm 2i)x} = e^{-3x} e^{\pm 2ix} = e^{-3x} (\cos 2x \pm i \sin 2x)$$

$\{e^{-3x}(\cos 2x + i \sin 2x), e^{-3x}(\cos 2x - i \sin 2x)\}$ is a complex basis of the 2-dimensional real soln space: complex conjugate pair

$$y = C_1 e^{-3x} e^{2ix} + C_2 e^{-3x} e^{-2ix}$$

$$= e^{-3x} (C_1 e^{2ix} + C_2 e^{-2ix}) = \bar{y} \text{ require be real}$$

$$= \bar{C}_1 e^{-2ix} + \bar{C}_2 e^{2ix} \rightarrow C_2 = \bar{C}_1, C_1 = \bar{C}_2 \text{ (same)}$$

$$= e^{-3x} \left(\underbrace{C_1 e^{2ix} + \bar{C}_1 e^{-2ix}}_{= 2 \operatorname{Re}(C_1 e^{2ix})} \right) \leftarrow \begin{cases} z = x + iy \\ \bar{z} = x - iy \\ z + \bar{z} = 2x \\ x = \operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}) \end{cases}$$

$$= \operatorname{Re}(2C_1 e^{2ix})$$

$$\text{let } 2C_1 = c_1 - ic_2$$

$$= e^{-3x} \operatorname{Re}((c_1 - ic_2)(\cos 2x + i \sin 2x))$$

$$= e^{-3x} \operatorname{Re}(c_1 \cos 2x + c_2 \sin 2x + (-c_2 \cos 2x + c_1 \sin 2x))$$

$$= \boxed{e^{-3x} (c_1 \cos 2x + c_2 \sin 2x)}$$

$\{e^{-3x} \cos 2x, e^{-3x} \sin 2x\}$ are an explicitly real basis of the soln space.

[just new lin. ind. linear combinations of the two complex exponentials]

Conclusion

Replace the complex conjugate exponentials by their real and imaginary parts to get a basis of the soln space. when complex root pairs come from the characteristic eqn.

5.3a) Higher order linear DE's and complex exponentials (2)

Impose initial conditions: $y(0) = 2, y'(0) = -4$

$$y = e^{-3x} (C_1 \cos 2x + C_2 \sin 2x)$$

$$y' = -3e^{-3x} (C_1 \cos 2x + C_2 \sin 2x) + e^{-3x} (-2C_1 \sin 2x + 2C_2 \cos 2x)$$

$$y(0) = C_1 \stackrel{= 2}{\underset{\downarrow}{=}}$$

$$y'(0) = -3C_1 + 2C_2 = -4 \rightarrow C_2 = \frac{1}{2}(-4 + 3C_1) = \frac{1}{2}(-4 + 6) = 1$$

so $(C_1, C_2) = (2, 1)$ and

$$y = e^{-3x} (2 \cos 2x + \sin 2x)$$

Summary

For a pair of roots $r_{\pm} = p \pm iq$ of the characteristic equation

$$e^{r_{\pm}x} = e^{(p \pm iq)x} = e^{px} e^{\pm iqx} = e^{px} (\cos qx \pm i \sin qx)$$

$\text{Re}(r_{\pm}) \rightarrow$ real exponential factor

$\text{Im}(r_{\pm}) \rightarrow$ frequency coefficient of cosine/sine factors

The real and imaginary parts of these two complex solutions of the DE are new independent linear combinations of solutions, so are also solutions:

$$\{e^{px} \cos qx, e^{px} \sin qx\} \text{ new basis for soln space}$$

$$y = C_1 e^{px} \cos qx + C_2 e^{px} \sin qx$$

$$= e^{px} (C_1 \cos qx + C_2 \sin qx) \quad \begin{matrix} \text{factored form.} \\ \text{(general soln)} \end{matrix}$$

coefficients determined
by initial conditions

5.3a) Higher order linear DEs and complex exponentials (3)

Consider n th order constant coefficient linear homogeneous DEs:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0 \leftarrow \text{substitute } y = e^{rx}$$

$$r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0 \quad \text{characteristic eqn}$$

$(r - r_1) \dots (r - r_n) = 0$ real polynomial factors into n linear factors with n roots r_1, \dots, r_n : either real or complex conjugate pairs

Suppose all n roots are distinct

Each real root leads to a soln $y = e^{rx}$ (real exponential).

Each complex conjugate pair of roots leads to

$$r_{\pm} = p \pm iq \rightarrow y = e^{(p \pm iq)x} = e^{px}(\cos qx \pm i \sin qx)$$

BUT replace complex solns by their independent real and imaginary parts:

$$y = e^{px} \cos qx, e^{px} \sin qx \quad 2 \text{ solns?}$$

The set of all such solns consists of n linearly ind. functions and forms a basis of the soln space.

general soln = arbitrary linear combination of these n functions

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4

example

$$\begin{cases} y^{(4)} - 16y = 0 \\ y(0) = 4, y'(0) = 0, y''(0) = 0, y'''(0) = 0 \end{cases}$$

$$y = e^{rx} \rightarrow (r^4 - 16)e^{rx} = 0 \rightarrow r^4 - 16 = 0$$

$$\text{but } r^4 - 16 = \underbrace{(r^2 - 4)}_{(r-2)(r+2)} \underbrace{(r^2 + 4)}_{\leftarrow} \quad \begin{array}{c} a^2 - b^2 = (a-b)(a+b) \\ \downarrow \qquad \qquad \qquad \downarrow \end{array} \quad r = \pm 2i$$

$$= (r-2)(r+2)(r-2i)(r+2i)$$

$$r = 2, -2, 2i, -2i$$

$$e^{rx} = e^{2x}, e^{-2x}, \underbrace{e^{2ix}, e^{-2ix}}$$

$$\therefore \underbrace{\cos 2x \pm i \sin 2x}$$

replace by $\cos 2x, \sin 2x$

Arbitrary linear combination:

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

gen. soln.

impose initial conditions:

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - 2c_3 \sin 2x + 2c_4 \cos 2x$$

$$y'' = 4c_1 e^{2x} + 4c_2 e^{-2x} - 4c_3 \cos 2x - 4c_4 \sin 2x$$

$$y''' = 8c_1 e^{2x} - 8c_2 e^{-2x} + 8c_3 \sin 2x - 8c_4 \cos 2x$$

$$y(0) = c_1 + c_2 + c_3 = 4$$

$$y'(0) = 2c_1 - 2c_2 + 2c_4 = 0$$

$$y''(0) = 4c_1 + 4c_2 - 4c_3 = 0$$

$$y'''(0) = 8c_1 - 8c_2 - 8c_4 = 0$$

$$\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \\ y'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$W(e^{2x}, e^{-2x}, \cos 2x, \sin 2x)|_{x=0} \rightarrow \text{invert}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 8 & 1 & 2 & 1 \\ 8 & -4 & 2 & -1 \\ 16 & 0 & -4 & 0 \\ 0 & 8 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 6 \end{bmatrix}$$

$$\text{so } y = e^{2x} + e^{-2x} + 2\cos 2x$$

textbook example 7

$$y''' + y' - 10y = 0 \xrightarrow{y=e^{rx}} (r^3 + r - 10)e^{rx} = 0$$

$$r^3 + r - 10 = 0 \xrightarrow{\text{Maple}} r = 2, -1 \pm 2i$$

$$e^{rx} = e^{2x}, e^{\underbrace{(-1 \pm 2i)x}_{\substack{= e^{-x}(\cos 2x \pm i \sin 2x)}}} = e^{-x} (\cos 2x \pm i \sin 2x)$$

$$\text{so } y = c_1 e^{2x} + e^{-x} (c_2 \cos 2x + c_3 \sin 2x)$$

textbook exercise 30

$$y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$$

$$\xrightarrow{y=e^{rx}} r^4 - r^3 + r^2 - 3r - 6 = 0$$

$$\hookrightarrow r = 2, -1, \pm i\sqrt{3}$$

$$e^{rx} = e^{2x}, e^{-x}, e^{\pm i\sqrt{3}x} = \cos \sqrt{3}x \pm i \sin \sqrt{3}x$$

$$\hookrightarrow \cos \sqrt{3}x, \sin \sqrt{3}x \quad (\text{real})$$

$$\text{so } y = c_1 e^{2x} + c_2 e^{-x} + c_3 \cos \sqrt{3}x + c_4 \sin \sqrt{3}x$$

student exercise:

$$y^{(4)} - y^{(3)} - y'' - y' - 2y = 0,$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 30$$

Find roots (Maple).

Write out general soln.

Solve initial conditions using inverse Wronskian matrix.

Backsub into y.