

5.1b

Constant coefficient linear homogeneous 2nd order DEs

①

We specialize to the constant coefficient case, and homogeneous DEs.

Revisit 1st order case:

$$\frac{dy}{dx} - ky = 0 \rightarrow \frac{dy}{dx} = ky \rightarrow y = Ce^{kx}$$

(perhaps) reasonable to expect that higher order case have exponential solns (with "indsight", "educated guess")

Suppose we "guess" a trial exponential soln with rate factor to be

determined: $y = e^{rx}$ Backsub into the DE:

$$\frac{dy}{dx} = re^{rx} \quad \frac{dy}{dx} - ky = 0 \rightarrow re^{rx} - ke^{rx} = 0 \\ (r-k)e^{rx} = 0$$

$$\begin{cases} r-k=0 \\ r=k \end{cases}$$

$e^{rx} = e^{rx}$ is a soln.

This generalizes to higher order.

2nd order (constant coeff linear hom.) case

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

$$\text{trial soln: } c \left[y = e^{rx} \right] \quad \left\{ \begin{array}{l} \frac{d^n y}{dx^n} = r^n e^{rx} \\ \frac{d^n y}{dx^n} = r^n y \end{array} \right.$$

$$\begin{array}{l} b \left[\frac{dy}{dx} = re^{rx} \right] \\ a \left[\frac{d^2y}{dx^2} = r^2 e^{rx} \right] \end{array}$$

The derivatives are converted to powers of r when evaluated on e^{rx}

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = ar^2 e^{rx} + bre^{rx} + ce^{rx} \\ = (ar^2 + br + c) e^{rx} = 0$$

$$ar^2 + br + c = 0$$

Solve with quadratic formula to get the allowed rate factors which do the job.

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$$\text{So } y = e^{rx} \rightarrow ar^2 + br + c = 0 \\ \rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \equiv r_{\pm}$$

Three cases

$b^2 - 4ac > 0$	(1) 2 real roots (distinct)	$r_1, r_2 = r_-, r_+$	2 distinct exponentials
$b^2 - 4ac = 0$	(2) 1 real root	$r_1 = r_+ = r_-$	1 exponential
$b^2 - 4ac < 0$	(3) 0 real roots	$r_+ = \bar{r}_-$	complex conjugate exponentials!

$$r_{\pm} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

Examples

$$(1) r_1 \neq r_2: y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad \text{linear combination of 2 distinct real exponentials}$$

Ex: $2y'' - 7y' + 3y = 0 \rightarrow 2r^2 e^{rx} - 7r e^{rx} + 3e^{rx} = 0$
 $(2r^2 - 7r + 3) e^{rx} = 0$

$$\begin{aligned} y &= e^{rx} \\ y' &= r e^{rx} \\ y'' &= r^2 e^{rx} \end{aligned}$$

$$2r^2 - 7r + 3 = 0 \\ r = \frac{7 \pm \sqrt{49 - 4(2)(3)}}{2(2)} = \frac{7 \pm \sqrt{25}}{4} = \frac{7 \pm 5}{4}$$

$$= \frac{12}{4}, \frac{2}{4} = 3, \frac{1}{2}$$

$$e^{rx} = e^{3x}, e^{x/2}$$

$$y = C_1 e^{3x} + C_2 e^{x/2}$$

Ex: $y'' + 2y' = 0 \rightarrow r^2 e^{rx} + 2r e^{rx} = 0$
 $\underbrace{(r^2 + 2r)}_{r(r+2)} e^{rx} = 0$

$$y = e^{rx} \uparrow$$

$$r(r+2) = 0 \rightarrow r = 0, -2$$

$$e^{rx} = e^{0x}, e^{-2x} = 1, e^{-2x}$$

$$\begin{aligned} y &= C_1(1) + C_2 e^{-2x} \\ &= C_1 + C_2 e^{-2x} \end{aligned}$$

$$(2) r_1 = r_2: y'' + 2y' + y = 0 \rightarrow \underbrace{(r^2 + 2r + 1)}_{(r+1)^2} e^{rx} = 0$$

$$y = e^{rx} \uparrow$$

$$(r+1)^2 = 0 \rightarrow r = -1$$

$e^{rx} = e^{-x}$ but falls short, need 2 ind. functions!!

Guessing falls short, we have to work harder!

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Repeated roots case

Consider the general case by working backwards: $r = r_0$

$$(r - r_0)^2 = 0 \rightarrow r^2 - 2r_0r + r_0^2 = 0 \Leftrightarrow y'' - 2r_0y' + r_0^2y = 0$$

$$\text{Trial soln: } y = e^{rx} \rightarrow r^2 e^{rx} - 2r_0 r e^{rx} + r_0^2 e^{rx} = 0$$

$$(r^2 - 2r_0 r + r_0^2) e^{rx} = 0$$

$$r^2 - 2r_0 r + r_0^2 = 0 \rightarrow (r - r_0)^2 = 0 \rightarrow r = r_0$$

so $y = e^{r_0 x}$ is g soln, we need another independent soln.

We change the dependent variable

from y to $y = ue^{r_0 x}$ to get a DE for u :

Backsubstitute to derive new DE.

$$\begin{aligned} &+ r_0^2 [y = ue^{r_0 x}] \\ &- 2r_0 [y' = u'e^{r_0 x} + ur_0 e^{r_0 x}] \\ &\downarrow [y'' = u''e^{r_0 x} + \underbrace{u'r_0 e^{r_0 x}}_{2r_0 u'e^{r_0 x}} + ur_0^2 e^{r_0 x}] \end{aligned}$$

$$y'' - 2r_0 y' + r_0^2 y = \left[\begin{array}{l} r_0^2 u \\ -2r_0 u' - 2r_0^2 u \\ + u'' + 2r_0 u' + r_0^2 u \end{array} \right] e^{r_0 x} = u'' e^{r_0 x} = 0$$

$$\begin{cases} u'' = 0 \\ u = c_1 + c_2 x \\ y = ue^{r_0 x} = (c_1 + c_2 x)e^{r_0 x} \\ = c_1 e^{r_0 x} + c_2 x e^{r_0 x} \end{cases}$$

Conclusion.

$$y = (c_1 + c_2 x) e^{r_0 x}$$

$$\text{basis: } \{e^{r_0 x}, x e^{r_0 x}\}$$

additional ind.
soln!

Return to example.

5.1b) Constant coefficient linear homogeneous 2nd order DEs (4)

Ex $y'' + 2y' + y = 0 \xrightarrow{y=e^{rx}}$ $r^2 + 2r + 1 = (r+1)^2 = 0 \rightarrow r = -1$
 $y = e^{-x}, \cdot X \theta^{-x}$
 $y = (C_1 + C_2 x) e^{-x}$

Add I.C.S:

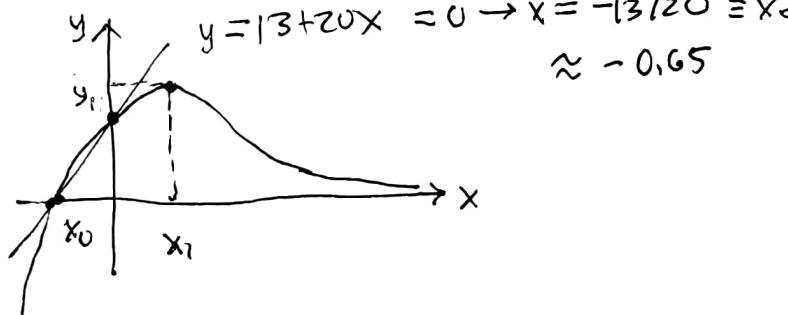
$$y(0) = 13, \quad y'(0) = -7 \quad (\text{cooked numbers!})$$

$$y' = C_2 e^{-x} + (C_1 + C_2 x) e^{-x}(-1) = (C_2 - C_1 - C_2 x) e^{-x}$$

$$\begin{aligned} y(0) &= (C_1 + 0) e^{-0} = C_1 = 13 \\ y'(0) &= (C_2 - C_1 - 0) e^{-0} = C_2 - C_1 = 7 \quad \downarrow \\ &\quad C_2 = 7 + C_1 = 7 + 13 = 20 \\ y &= (13 + 20x) e^{-x} \end{aligned}$$

But once we solve the IVP, we need to analyze the behavior of the solution.

1 global max



$$y = (13 + 20x) e^{-x}$$

$$\begin{aligned} y' &= 20e^{-x} + (13 + 20x)e^{-x}(-1) \\ &= (20 - 13 - 20x)e^{-x} \end{aligned}$$

$$\begin{aligned} &= (7 - 20x)e^{-x} = 0 \rightarrow x = \frac{7}{20} > 0 \\ &\quad \equiv x_1 \approx 0.35 \end{aligned}$$

$$y(x_1) = (13 + 20(\frac{7}{20})) e^{-\frac{7}{20}}$$

$$= (13 + 7) e^{-7/20}$$

$$= 20 e^{-7/20} \approx 14.094$$

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case(3): $r \pm$ complex

Ex. $y'' + y = 0 \rightarrow (r^2 + 1) e^{rx} = 0$

$$y = e^{rx}$$

$$r^2 + 1 = 0$$

$$r^2 = -1$$

$$r = \pm \sqrt{-1} = \pm i$$

require be real!

$$e^{rx} = e^{\pm ix}$$

$$y = C_1 e^{ix} + C_2 e^{-ix}$$

$$\downarrow \\ = \bar{y}$$

$$= \bar{C}_1 e^{-ix} + \bar{C}_2 e^{ix}$$

$$\rightarrow C_2 = \bar{C}_1 \text{ complex conjugates}$$

$$y = C_1 e^{ix} + \bar{C}_1 e^{-ix}$$

$$= 2 \operatorname{Re}(C_1 e^{ix})$$

we need to revisit complex arithmetic!

Next time.