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Second order linear DEs: intro

①

The jump from 1st order to 2nd order linear DEs is the model for all higher order linear DEs. Recall:

$$\text{linear combination } \left\{ y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n} \right\} = f(x)$$

homogeneous part nonhomogeneous part

Before we dive into the case $n=2$:

Motivating Example (linear homogeneous case)

IVP: $\begin{cases} y'' + 9y = 0 & (\text{DE}) \\ y(0) = 2, y'(0) = 3 & (\text{initial or I.C.s}) \end{cases}$

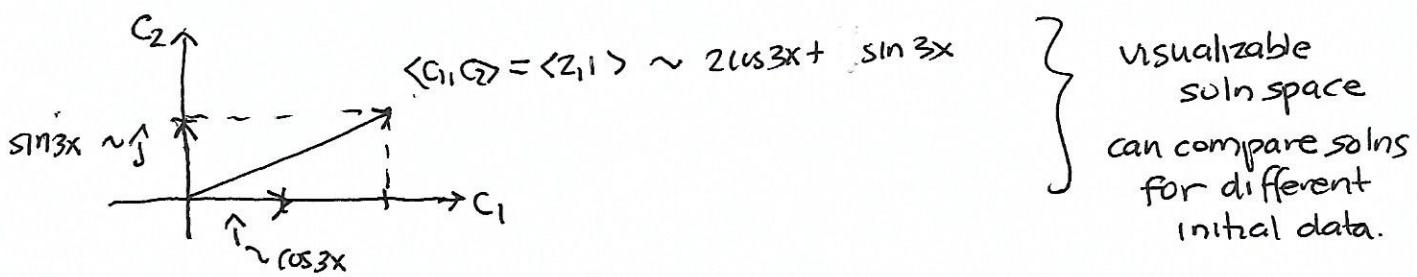
soln: $y = C_1 \cos 3x + C_2 \sin 3x \leftrightarrow \{ \cos 3x, \sin 3x \}$ is a basis of soln space
 (C_1, C_2) coords on this subspace

impose i'nts: $y' = -3C_1 \sin 3x + 3C_2 \cos 3x$

$y(0) = C_1 \cos(0) + C_2 \sin(0) = 2 \rightarrow C_1 = 2$ ← soln of I.Cs
 $y'(0) = -3C_1 \sin(0) + 3C_2 \cos(0) = 3 \rightarrow 3C_2 = 3 \rightarrow C_2 = 1$ ← not of DE!

matrix form $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

backsub $\boxed{y = 2 \cos 3x + \sin 3x}$



special case of $y'' + \omega^2 y = 0, \omega > 0$ "frequency"

soln: $\boxed{y = C_1 \cos \omega x + C_2 \sin \omega x}$ derive later, now check

0 $[y' = -C_1 \omega \sin \omega x + C_2 \omega \cos \omega x]$

1 $[y'' = -C_1 \omega^2 \cos \omega x - C_2 \omega^2 \sin \omega x]$

$$y'' + \omega^2 y = C_1 \omega^2 \cos \omega x + C_2 \omega^2 \sin \omega x = 0 \quad \checkmark$$

$\cancel{-C_1 \omega^2 \cos \omega x}$ $\cancel{-C_2 \omega^2 \sin \omega x}$

Memorize soln just like: $y' = ky \Leftrightarrow y = C e^{kx}$

We need it over and over again

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general 2nd order DE: $G(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$ too hard!

↓

linear case: $A(x) \frac{d^2y}{dx^2} + B(x) \frac{dy}{dx} + C(x)y = F(x)$

divide thru

linear homogeneous part
y, y', y'' terms only

nonhomogeneous part
"driving term"

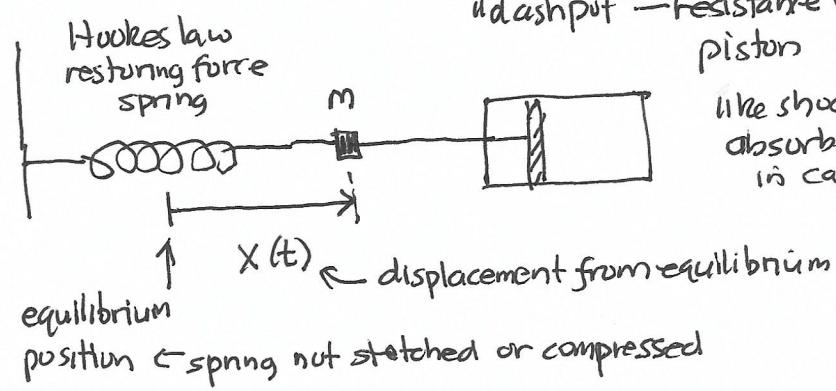
standard form: $1 \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = f(x) \rightarrow 0$ homogeneous case

Application "damped harmonic oscillator"

1d motion mechanics problem

x → t time

y → X displacement



eqn of motion

$$m \frac{d^2X}{dt^2} = -kX - c \frac{dX}{dt} + F(t)$$

(mass)(acc.) Hooke's law force frictional force (resistance) additional driving force applied to mass

↓

$$m \frac{d^2X}{dt^2} + c \frac{dX}{dt} + kX = F(t)$$

all coefficients ≥ 0
constants ($m \neq 0$)

↓

$$1 \frac{d^2X}{dt^2} + \left(\frac{c}{m}\right) \frac{dX}{dt} + \left(\frac{k}{m}\right) X = \frac{F(t)}{m} = f(t)$$

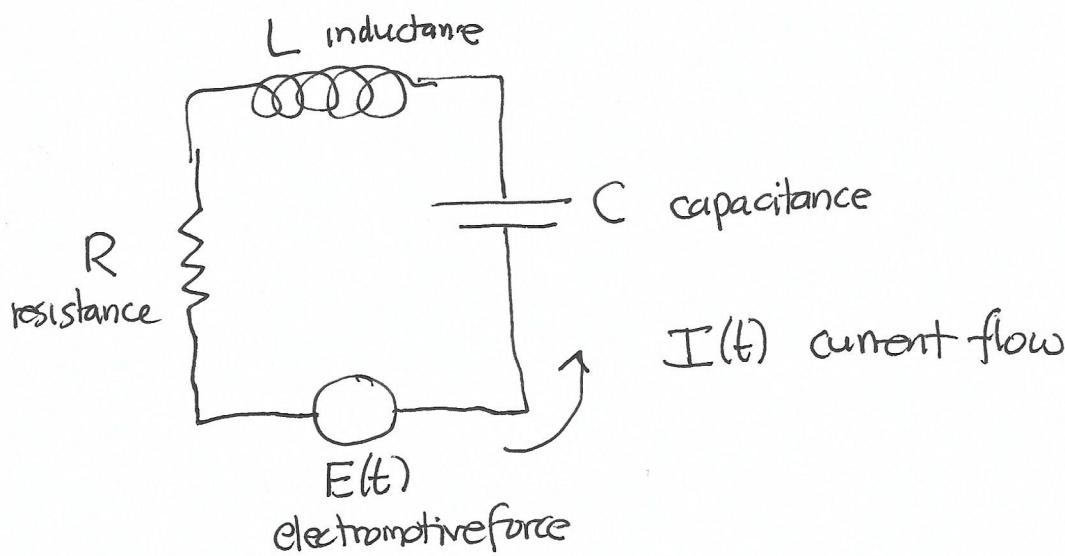
standard form.
only 2 parameters

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Another similar example Application : RLC Circuit



eqn of "motion":

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + C^{-1} I = E(t)$$

L | |] constants ≥ 0

Both applications have same form

$$a y'' + b y' + c y = F(x) \quad a, b, c \text{ same sign (if not 0)}$$

a b c F(x) constant coefficients

We can solve this mathematical problem, we can plug in parameters of the application.

In fact we are only interested in constant coefficient linear DES. (easy to solve, lots of applications)

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Linear superposition for linear homogeneous DE's

$$y'' + py' + qy = 0 \quad \text{soln space is closed under linear ops}$$

two solns: $y_1'' + py_1' + qy_1 = 0 \rightarrow c(y_1'' + py_1' + qy_1) = c(0)$
 $y_2'' + py_2' + qy_2 = 0 \quad cy_1'' + pc y_1' + qc y_1 = 0$
 $\underline{(cy_1)'' + p(cy_1)' + q(cy_1) = 0}$
 $(cy_1)'' + p(cy_1)' + q(cy_1) = 0$

$$\underbrace{y_1'' + y_2''}_{(y_1+y_2)''} + py_1' + py_2' + qy_1 + qy_2 = 0 + 0 \quad cy_1 \text{ also a soln}$$

$$(y_1+y_2)'' + p(y_1+y_2)' + q(q_1+q_2) = 0$$

y_1+y_2 also a soln

soln space is closed under linear combination
= subspace of ∞ -dim space of differentiable functions

We know there must be 2 arbitrary constants in every 2nd order
- DE soln so it is a 2-dim subspace

basis functions $\{y_1, y_2\}$

$$\text{gen. soln: } y = c_1 y_1 + c_2 y_2$$

Nonhomogeneous case

$$y_p'' + py_p' + qy_p = f \quad \text{particular soln } y_p \text{ (no parameters)}$$

$$y_h'' + py_h' + qy_h = 0 \quad \text{gen soln with 2 arb constants}$$

$$\underline{y_p'' + y_h'' + py_p' + py_h' + qy_p + qy_h = f + 0}$$

$$(y_p + y_h)'' + p(y_p + y_h)' + q(y_p + y_h) = f \quad \text{also a soln of nonhom DE!}$$

Suggests DIVIDE & CONQUER strategy:

First solve homogeneous case

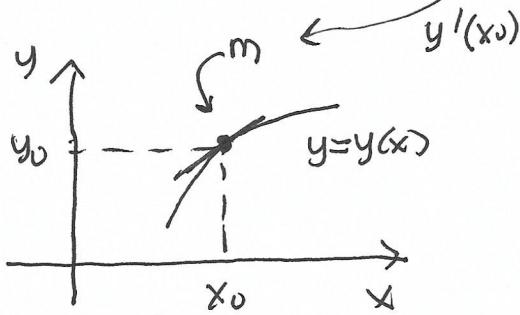
then consider solvable nonhomogeneous cases

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Visualize solns

$$\text{1st order} \left\{ \begin{array}{l} y' + p(x)y = q(x) \quad (\text{DE}) \\ \text{I.V.P: } \begin{cases} y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases} \end{array} \right.$$



pick initial point, DE determines slope

$$\begin{aligned} y &= e^{-\int p dx} \left[\int q e^{\int p dx} dx + C \right] \\ &= e^{-\int p dx} \underbrace{\int q e^{\int p dx} dx}_{y_p} + C e^{-\int p dx} \underbrace{e^{-\int p dx}}_{y_h} \\ &\text{I.C. determines } C \end{aligned}$$

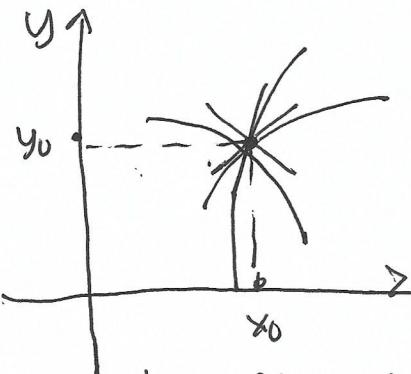
$$\begin{aligned} \text{2nd order} \left\{ \begin{array}{l} y'' + p(x)y' + q(x)y = f(x) \quad (\text{DE}) \\ \text{I.V.P: } \begin{cases} y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases} \end{array} \right. \end{aligned}$$

} I.C.s

$y = y_p + y_h$

$y_h = c_1 y_1 + c_2 y_2$

I.C.s determine arbitrary constants



pick initial point, pick slope
DE determines rest

} soln curves exit initial point
in all directions

But when does a soln exist?

THM As long as p, q fall continuous on some interval,
linearity guarantees a unique soln exists on the same interval.

(No worries as in nonlinear first order case)

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Solving the I.C.s is a system of linear equations

homogeneous case: $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = y_0$, $y'(x_0) = v_0$ solve: $y = c_1 y_1 + c_2 y_2$ impose initial conditions: $y' = c_1 y_1' + c_2 y_2'$

$$\begin{aligned}y(x_0) &= c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \\y'(x_0) &= c_1 y_1'(x_0) + c_2 y_2'(x_0) = v_0\end{aligned}$$

matrix form:

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}^{-1} \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

analogous
to change
of words
on soln space

$$\mathbf{B}^{-1} \mathbf{X} = \mathbf{Y}$$

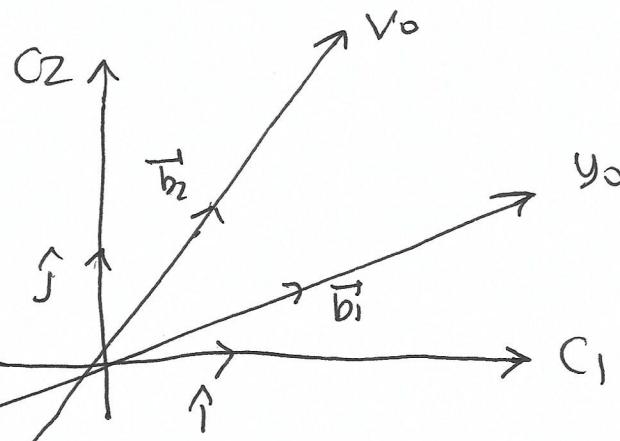
old
coords new
coords

$$\mathbf{X} = \mathbf{B} \mathbf{Y}$$

old new
↓

$$\langle b_1 | b_2 \rangle$$

inverse
matrix
defines
new
basis
vectors



\hat{i}, \hat{j} are coord vectors of y_1, y_2 (basis)

coord space of soln subspace

allows comparison of different initial data
solns for given DE

"Second visualization" of DE solns!

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Example (unit frequency)

$$y'' + y = 0 \rightarrow y = c_1 \cos x + c_2 \sin x$$

$$\hookrightarrow y' = -c_1 \sin x + c_2 \cos x$$

at $x=0$: $y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = y_0$

$$y'(0) = -c_1 \sin(0) + c_2 \cos(0) = c_2 = v_0$$

nothing to solve
arb constants
are the initial
data

matrix form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$

unit coeff
matrix!

{ $\cos x, \sin x$ } are the "natural basis"
for initial conditions at $x=0$.

But if $x \neq 0$, then what?

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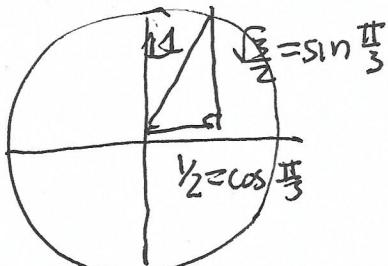
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At $x = \frac{\pi}{3}$

$$y\left(\frac{\pi}{3}\right) = c_1 \cos\left(\frac{\pi}{3}\right) + c_2 \sin\left(\frac{\pi}{3}\right) = y_0$$

$$y'\left(\frac{\pi}{3}\right) = -c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \cos\left(\frac{\pi}{3}\right) = v_0$$

values:



unit circle

but value don't matter,
true for any angle

$$\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ -\sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$$

$$\det: \cos^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{3}\right) = 1 ! \text{ easy.}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$$

$$= \begin{bmatrix} y_0 \cos\left(\frac{\pi}{3}\right) - v_0 \sin\left(\frac{\pi}{3}\right) \\ y_0 \sin\left(\frac{\pi}{3}\right) + v_0 \cos\left(\frac{\pi}{3}\right) \end{bmatrix}$$

soln of I.C.s
not of DE!
backsub !!

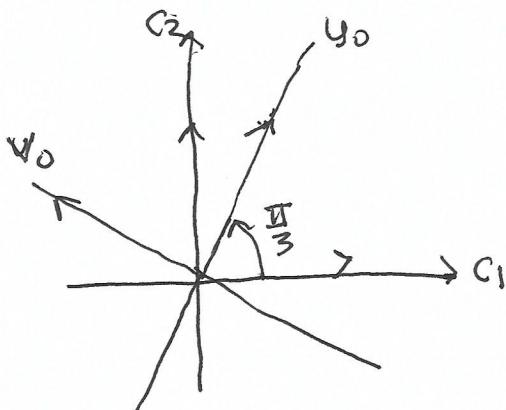
$$y = (y_0 \cos\left(\frac{\pi}{3}\right) - v_0 \sin\left(\frac{\pi}{3}\right)) \cos x + (y_0 \sin\left(\frac{\pi}{3}\right) + v_0 \cos\left(\frac{\pi}{3}\right)) \sin x$$

$$= y_0 (\cos x \cos\left(\frac{\pi}{3}\right) + \sin x \sin\left(\frac{\pi}{3}\right)) + v_0 (\sin x \cos\left(\frac{\pi}{3}\right) - \cos x \sin\left(\frac{\pi}{3}\right))$$

$$= y_0 \cos(x - \frac{\pi}{3}) + v_0 \sin(x - \frac{\pi}{3})$$

(subtraction identities!)

$\{\cos(x - \frac{\pi}{3}), \sin(x - \frac{\pi}{3})\}$
new basis solns

new coords on
soln spacesoln space picture
axes rotated by $\pi/3$!