

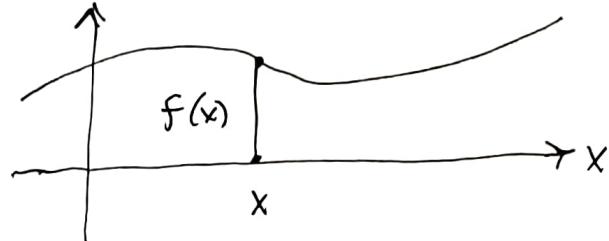
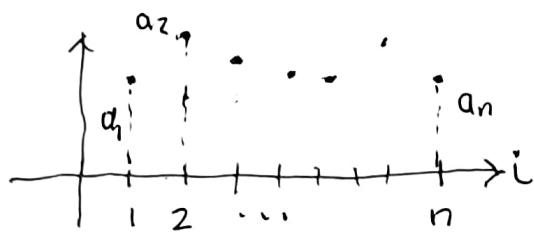
4.7

functions as "vectors"

①

A function $f(x)$ is like a vector whose components/coordinates are labeled by a continuous index $x \in \mathbb{R}$ instead of a discrete index $i = 1, \dots, n$

$$\vec{a} = \langle a_1, \dots, a_n \rangle \leftrightarrow (a_1, \dots, a_n)$$



In both cases corresponding "components" are added and all "components" simultaneously multiplied by numbers (scalars).

$$\vec{a} + \vec{b}$$

$$= \langle a_1, \dots, a_n \rangle$$

$$+ \langle b_1, \dots, b_n \rangle$$

$$= \langle a_1 + b_1, \dots, a_n + b_n \rangle$$

$$(f+g)(x)$$

$$= f(x) + g(x) \quad \text{"vector addition"}$$

$$(\vec{a} + \vec{b})_i = a_i + b_i$$

$$c\vec{a} = c \langle a_1, \dots, a_n \rangle$$

$$= \langle ca_1, \dots, ca_n \rangle$$

$$(c\vec{a})_i = c a_i$$

$$(cf)(x) = c f(x) \quad \text{"scalar multiplication"}$$

This makes the space of all realvalued functions of x into an ∞ -dimensional vector space

In particular differentiable functions are closed under these linearity properties

$$\frac{d}{dx}(f+g)(x) = \frac{df(x)}{dx} + \frac{dg(x)}{dx} \quad \text{"sum rule"}$$

$$\frac{d}{dx}(cf)(x) = c \frac{df(x)}{dx} \quad \text{"constant factor rule"}$$

so this is a linear subspace where we can consider DEs.

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functions as "vectors"

linear homogeneous conditions
on \mathbb{R}^n $A X = 0$

pick out subspaces:

linear combinations of $\{x_i\} = 0$

↓ solved by row
reduction

sln = arbitrary linear comb.
of basis vectors

$$\vec{x} = t_1 \vec{e}_1 + \dots + t_p \vec{e}_p$$

linear homogeneous DEs
pick out subspaces of the
co-dim space of differentiable
functions as "sln spaces"

linear combinations of
 $\{y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\} = 0$
(linear hom. DEs!)

sln = arbitrary linear comb
of "basis" functions

$$y = c_1 y_1 + \dots + c_n y_n$$

↑ ↑
must be linearly ind,
and span sln space!

sln techniques produce
slns but we need to
make sure we get a
basis for all slns.

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functions as "vectors"

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Example:

$$\int \left[\frac{d^3y}{dx^3} = 0 \right] dx \rightarrow \underbrace{\int \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) dx}_{\frac{dy}{dx}} = \int 0 dx = 0 + C_1$$

$$\int \left[\frac{d^2y}{dx^2} = C_1 \right] dx \rightarrow \underbrace{\int \frac{d}{dx} \left(\frac{dy}{dx} \right) dx}_{\frac{dy}{dx}} = \int C_1 dx = C_1 x + C_2$$

$$\int \left[\frac{dy}{dx} = C_1 x + C_2 \right] dx \rightarrow \int \frac{dy}{dx} dx = \int C_1 x + C_2 dx = C_1 \frac{x^2}{2} + C_2 x + C_3$$

$$y = \frac{1}{2} C_1 x^2 + C_2 x + C_3 = ax^2 + bx + c \quad (\text{standard!})$$

↑ useless, redefine constants $= c + bx + ax^2$ ← reverse order
so can generalize to higher order polynomials:

Define $Q = \{q(x) \mid q \text{ is at most quadratic in } x\}$
allow $a=0$ (not quadratic!)

$$= \{ax^2 + bx + c \mid (a, b, c) \in \mathbb{R}^3\}$$

This is the soln space of the DE. It is a 3-dim subspace of the space of differentiable functions of x .

Note: $q = c + bx + ax^2 = c(1) + b(x) + a(x^2)$
is an arbitrary linear combination of $\{1, x, x^2\}$
so $Q = \text{span}(\{1, x, x^2\})$

and if $c + bx + ax^2 = 0$ (for all x) then $(c, b, a) = (0, 0, 0)$

so linearly independent.

Thus $\{1, x, x^2\}$ are a basis of the soln space and its (ordered) coefficients are the coordinates of q .

$$a_0 + a_1 x + \dots + a_n x^n$$

↓ Taylor

$$a_0 + \dots + a_n x^n + \dots$$

Infinite series

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functions as "vectors"

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$$q = c + bx + ax^2 \leftrightarrow \vec{q} = \langle c, b, a \rangle \text{ coordinate vector}$$

The function operations are exactly mirrored by the coordinate vector operations:

$$q_1 = 1 + 2x + 3x^2$$

$$q_2 = 2 - x + x^2$$

$$\begin{aligned} q_1 + q_2 &= (1+2) + (2-1)x + (3+1)x^2 \\ &= 3 + x + 4x^2 \end{aligned}$$

$$\begin{aligned} 2q_1 &= 2(1 + 2x + 3x^2) \\ &= 2 + 4x + 6x^2 \end{aligned}$$

$$\vec{q}_1 = \langle 1, 2, 3 \rangle$$

$$\vec{q}_2 = \langle 2, -1, 1 \rangle$$

$$\begin{aligned} \vec{q}_1 + \vec{q}_2 &= \langle 1+2, 2-1, 3+1 \rangle \\ &= \langle 3, 1, 4 \rangle \end{aligned}$$

$$\begin{aligned} 2\vec{q}_1 &= 2\langle 1, 2, 3 \rangle \\ &= \langle 2, 4, 6 \rangle \end{aligned}$$

Thus all the linear operations

on the abstract space of

functions can be done instead on the coordinate space \mathbb{R}^3 ,
the points of which all represent up to quadratic functions.

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functions as "vectors"

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linear change of coordinates / basis

It is natural to consider a change of basis on \mathbb{Q} . That is exactly what Taylor polynomials do (Calc 2!).

$$\begin{aligned} C + bx + ax^2 &= C_1 + B(x-1) + A(x-1)^2 \\ &= C_1 + BX - B + Ax^2 - 2Ax + A \\ &= (C_1 - B + A) + (B - 2A)x + (A)x^2 \end{aligned}$$

$$\begin{array}{lcl} C = C_1 - B + A & \leftrightarrow & \begin{bmatrix} C \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ B \\ A \end{bmatrix} \\ b = B - 2A & & \\ a = A & & \end{array}$$

$$\begin{array}{ccc} x & = & B \\ \text{old} & & \text{new} \end{array} \quad \begin{array}{ccc} y & & \\ \text{new} & & \end{array} \quad \begin{array}{l} \text{old: } \vec{x} = \langle C, b, a \rangle \\ \text{new: } \vec{y} = \langle C_1, B, A \rangle \end{array}$$

↓ invert using inverse matrix, but upper triangular coeff matrix allows backsub solution

$$\begin{array}{lcl} C_1 - B + A = C & \longrightarrow & C_1 = C + (B - 2A) - a \\ B - 2A = b & \longrightarrow & B = b + 2a \\ A = a & \uparrow & \end{array} \quad \begin{array}{l} C = C + (B - 2a) - a \\ = C + b + a \end{array}$$

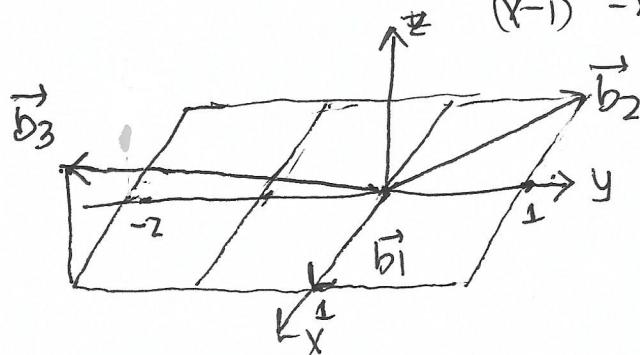
$$\begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ B \\ A \end{bmatrix}$$

$$y = B^{-1} x$$

$$\begin{array}{l} B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\ = \langle b_1 | b_2 | b_3 \rangle \end{array}$$

columns of B represent old coords of new basis vectors

$$\begin{array}{c} 1 \\ X-1 \\ (X-1)^2 = X^2 - 2X + 1 \end{array} \quad \begin{array}{l} \leftrightarrow \vec{b}_1 = \langle 1, 0, 0 \rangle \\ \leftrightarrow \vec{b}_2 = \langle -1, 1, 0 \rangle \\ \leftrightarrow \vec{b}_3 = \langle 1, -2, 1 \rangle \end{array}$$



concrete picture of the vectors corresponding to the solution functions!
 $\{\hat{i}, \hat{j}, \hat{k}\}$ correspond to original basis