

4.4 Bases for \mathbb{R}^n

1

A **basis** for any vectorspace V is a (finite) set of vectors satisfying 2 conditions:

- 1) The vectors are linearly independent (no redundancy)
 - 2) They span the entire space.

This gives us a unique way to express every vector in the space as a linear combination of the basis vectors. By picking a specific order of the vectors, we can associate the coefficients with the values of the coordinates of the vectors with respect to this basis (equivalently "components" of vectors with respect to the basis).

For \mathbb{R}^n a basis $\{\vec{b}_1, \dots, \vec{b}_n\}$ means

$$\langle x_1, \dots, x_n \rangle = y_1 \vec{b}_1 + \dots + y_n \vec{b}_n \quad \leftarrow \text{any vector can be expressed as a unique}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \langle b_1 | \dots | b_n \rangle \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

linear combination of
these linearly ind. vectors

$$\underbrace{x = By}_{\text{"coordinate transformation"}} \quad \hookrightarrow \vec{y} = \langle y_1, \dots, y_n \rangle \quad \text{"new coordinate vector"} \\ \vec{x} = \langle x_1, \dots, x_n \rangle \quad \text{"old coordinate vector"}$$

Linear independence means $\det(B) \neq 0$ so B^{-1} exists.

The new coordinates/components are obtained by :

$$\rightarrow B^{-1}[x = By] \rightarrow B^{-1}x = y \rightarrow y = B^{-1}x \quad \text{"inverse coordinate transformation"}$$

points
or
vectors?

All the matrix manipulations are done with vectors, but the vectors are points in \mathbb{R}^n so

(x_1, \dots, x_n) old coordinates (y_1, \dots, y_n) new coordinates	$\langle x_1, \dots, x_n \rangle$ old components $\langle y_1, \dots, y_n \rangle$ new components
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so we will slough back and forth with point versus vector terminology!

4.4 Bases of \mathbb{R}^n

(2)

Example $\begin{aligned}\vec{b}_1 &= \langle 1, 2, 3 \rangle \\ \vec{b}_2 &= \langle 2, 3, 1 \rangle \\ \vec{b}_3 &= \langle 3, 1, 2 \rangle\end{aligned}$ $\left\{ \right. \quad \left. \begin{matrix} B = \langle b_1 | b_2 | b_3 \rangle = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \right.$

$\det(B) = -18$ so invertible: $B^{-1} = \langle b_1, b_2, b_3 \rangle = \frac{1}{18} \begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$

$\langle x_1, x_2, x_3 \rangle = y_1 \langle 1, 2, 3 \rangle + y_2 \langle 2, 3, 1 \rangle + y_3 \langle 3, 1, 2 \rangle$ vector form

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{Find old coords of point}} = y_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} y_1 + 2y_2 + 3y_3 \\ 2y_1 + 3y_2 + y_3 \\ 3y_1 + y_2 + 2y_3 \end{bmatrix} \leftrightarrow \begin{aligned} x_1 &= y_1 + 2y_2 + 3y_3 \\ x_2 &= 2y_1 + 3y_2 + y_3 \\ x_3 &= 3y_1 + y_2 + 2y_3 \end{aligned}$$

"linear coordinate transformation"

Scalar form

$x = B y$ matrix form

Go reverse direction: "invert" the coord transformation.

$\rightarrow B^{-1}(x = By) \rightarrow B^{-1}x = y$

$$y = B^{-1}x \leftrightarrow \begin{aligned} y_1 &= \frac{1}{18}(-5x_1 + x_2 + 7x_3) \\ y_2 &= \frac{1}{18}(x_1 + 7x_2 - 5x_3) \\ y_3 &= \frac{1}{18}(7x_1 - 5x_2 + x_3) \end{aligned}$$

inverse coordinate transformation

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\text{Basis vectors}} = x_1 \underbrace{\begin{bmatrix} -5/18 \\ 1/18 \\ 7/18 \end{bmatrix}}_{b_1} + x_2 \underbrace{\begin{bmatrix} 1/18 \\ 7/18 \\ 5/18 \end{bmatrix}}_{b_2} + x_3 \underbrace{\begin{bmatrix} 7/18 \\ -5/18 \\ 1/18 \end{bmatrix}}_{b_3}$$

$B^{-1} = \langle b_1 | b_2 | b_3 \rangle$

cols are new coords of old basis vectors

like

$B = \langle b_1 | b_2 | b_3 \rangle$

cols are old coords of new basis vectors

example: new coords of $\langle 1, 0, 0 \rangle$:

$$-\frac{5}{18} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{18} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \frac{7}{18} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -5 + 2 + 21 \\ 10 + 3 + 7 \\ -15 + 1 + 14 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

"relativity"

4.4

Bases of \mathbb{R}^n

3

If $\langle y_1, y_2, y_3 \rangle = \langle 2, 1, -1 \rangle$, what is $\langle x_1, x_2, x_3 \rangle$?

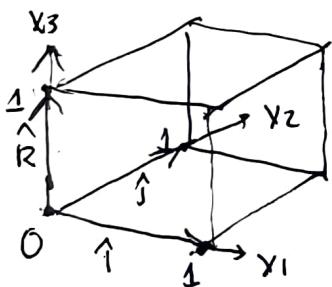
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+2-3 \\ 4+3+1 \\ 6+1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$$

If $\langle x_1, x_2, x_3 \rangle = \langle 1, 6, 5 \rangle$, what is $\langle y_1, y_2, y_3 \rangle$?

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -5+6+35 \\ 1+42-30 \\ 7-30+5 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 36 \\ 18 \\ -18 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Of course the "coordinate grid" of the new coords is on an expanded scale so the points with integer coords in both systems are separated!

see handout for 3d diagrams of "coordinate grid"

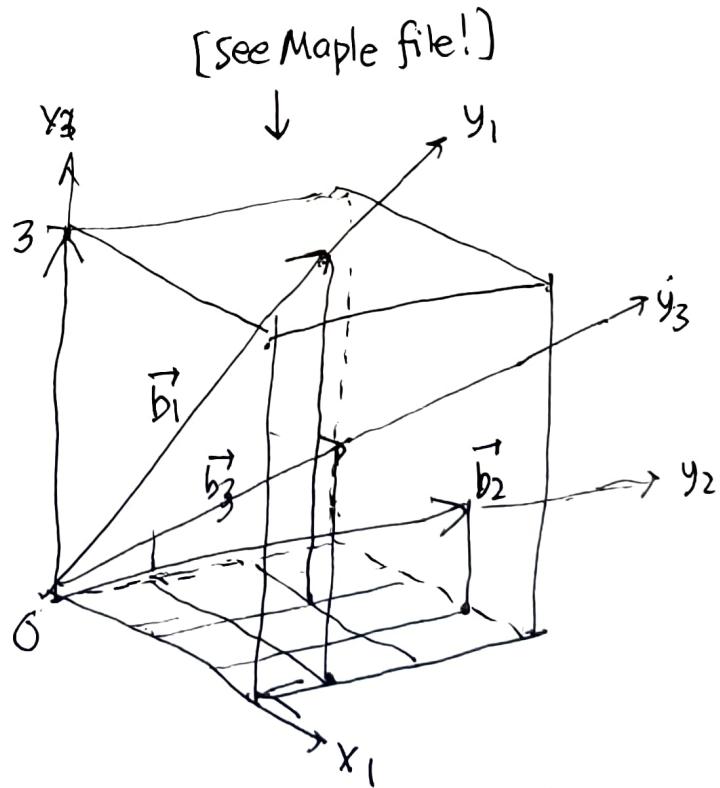


unit coordinate cube
in usual rectangular
coords on \mathbb{R}^3

edges are standard
basis vectors

$$0 \leq x_i \leq 1$$

Building blocks
for coordinate
"grids"



unit coordinate "cube"

for new coords: only 3 edges
 $0 \leq y_i \leq 1$ shown here

4.4

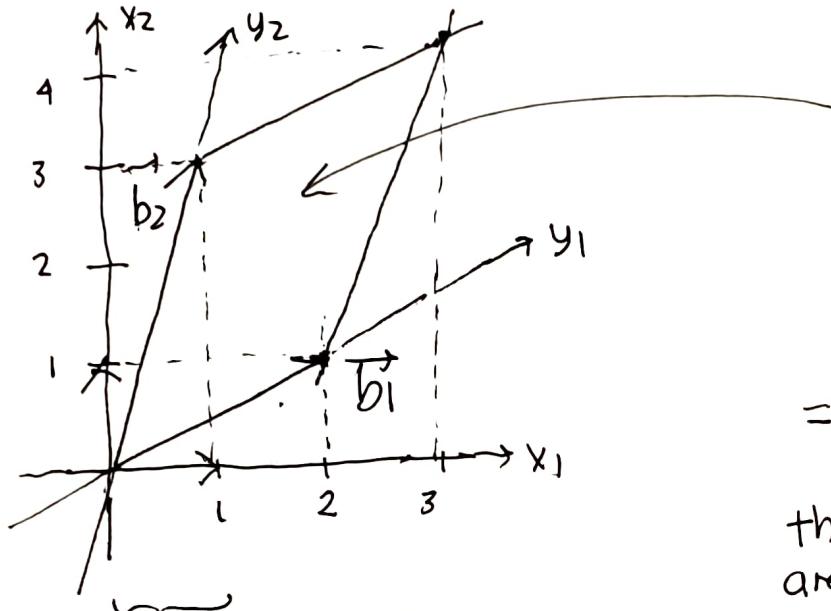
Bases of \mathbb{R}^n

4

All this is easier in 2 dimensions! No need for 3d perspective on 2d sheets of paper / screens!

New basis $\vec{b}_1 = \langle 2, 1 \rangle, \vec{b}_2 = \langle 1, 3 \rangle$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \rightarrow B^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$$



$0 \leq x_1 \leq 1$
 $0 \leq x_2 \leq 1$
 unit coord square
 has area

$$|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$0 \leq y_1 \leq 1$
 $0 \leq y_2 \leq 1$
 unit coord "square"
 (parallelogram as)
 (seen in old grid!)

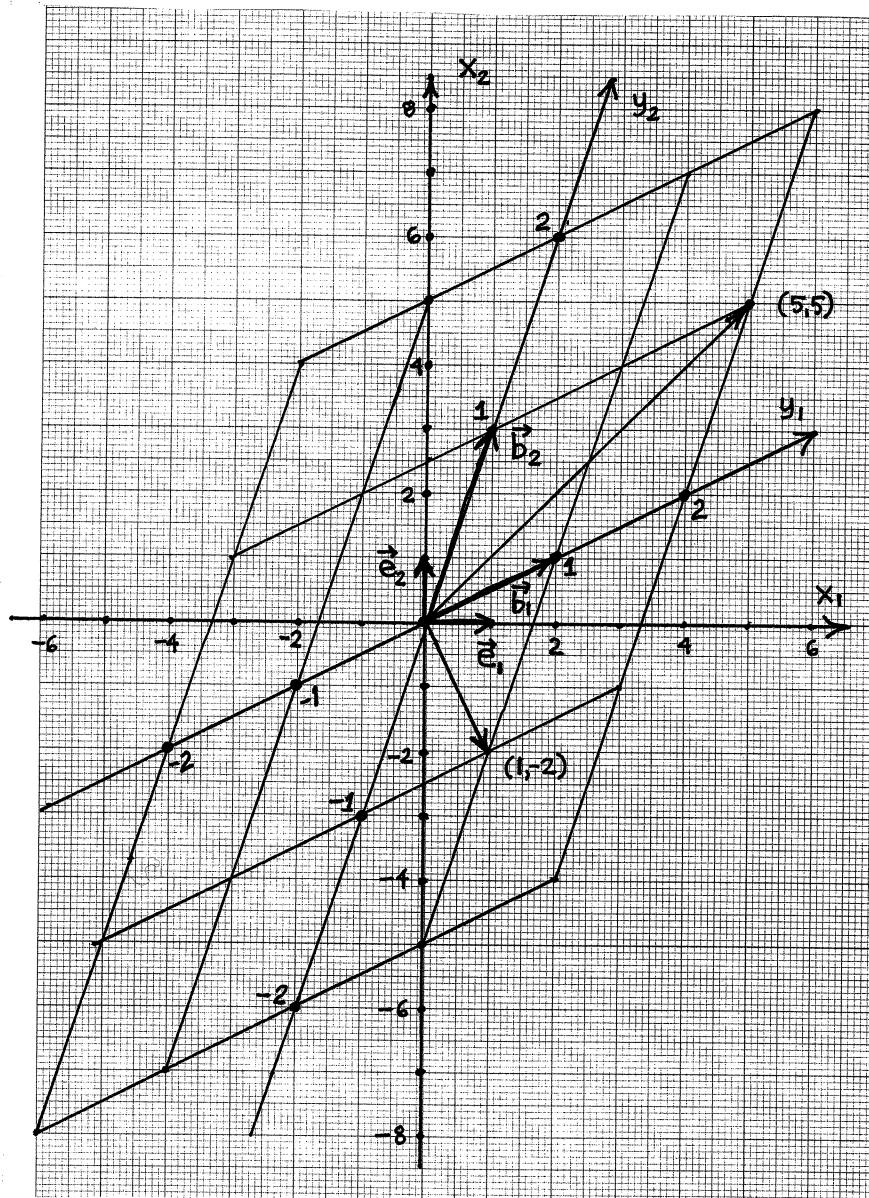
area
 $= |B| = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5$

the new grid expands areas by a factor of 5!

now replicate, "tiling" the plane with these parallelograms to make the new grid.

[see handout]

Nonstandard coordinates on \mathbb{R}^2



old basis:
 $\vec{e}_1 = \langle 1, 0 \rangle, \vec{e}_2 = \langle 0, 1 \rangle$
 new basis:
 $\vec{b}_1 = \langle 2, 1 \rangle, \vec{b}_2 = \langle 1, 3 \rangle$

basis changing matrix:
 $B = \text{augment } (\vec{b}_1, \vec{b}_2) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$
 $= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ cols = old coords of new basis vectors

inverse:
 $B^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$
 cols = new coords of old basis vectors

2 example vectors:

$$\begin{aligned} \langle 5, 5 \rangle &= 5\vec{e}_1 + 5\vec{e}_2 \\ &= 2\vec{b}_1 + 1\vec{b}_2 \end{aligned}$$

$$\begin{aligned} x_1 &= 5, x_2 = 5 \\ y_1 &= 2, y_2 = 1 \end{aligned}$$

$$\begin{aligned} \langle 1, -2 \rangle &= 1\vec{e}_1 - 2\vec{e}_2 \\ &= 1\vec{b}_1 - 1\vec{b}_2 \end{aligned}$$

$$\begin{aligned} x_1 &= 1, x_2 = -2 \\ y_1 &= 1, y_2 = -1 \end{aligned}$$

$$\begin{array}{l} \vec{e}_2 = (0, 1) \\ \vec{e}_1 = (1, 0) \\ \text{unit} \end{array}$$

The graph paper has the unit coordinate grid for the standard cartesian coordinates $\{x_1, x_2\}$ marked by bold line 1cm tickmarks (refined to 1mm tenth tickmarks).

The unit coordinate rectangle at the origin tiles the plane to make this grid.

The new coordinate grid is a tiling of the plane by the unit coordinate "rectangle" $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$ (actually a parallelogram) at the origin. The region shown is $-2 \leq y_1 \leq 2, -2 \leq y_2 \leq 2$.

The relationship between the two coordinate systems

$$\vec{X} = \langle x_1, x_2 \rangle = x_1 \vec{e}_1 + x_2 \vec{e}_2 = y_1 \vec{b}_1 + y_2 \vec{b}_2 = y_1 \langle 2, 1 \rangle + y_2 \langle 1, 3 \rangle$$

$$\text{or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{or } \begin{matrix} \vec{X} = B \vec{Y} \\ \text{"old"} \quad \text{"new"} \end{matrix} \leftrightarrow \begin{matrix} \vec{Y} = B^{-1} \vec{X} \\ \text{"new"} \quad \text{"old"} \end{matrix}$$

$$\text{or } \begin{aligned} x_1 &= 2y_1 + y_2 \\ x_2 &= y_1 + 3y_2 \end{aligned}$$

linear change
of coordinates

$$\begin{aligned} y_1 &= (3x_1 - x_2)/5 \\ y_2 &= (-x_1 + 2x_2)/5 \end{aligned}$$

4.4 Bases of \mathbb{R}^n

return to $A \times = \emptyset$

$m \times n$
matrix

✓ \hookrightarrow gen soln $x = t_1 e_1 + \dots + t_p e_p \in \mathbb{R}^n$

$\hookrightarrow \{e_1, \dots, e_p\}$

is a basis of the
soln space:
 p -plane thru origin
of \mathbb{R}^n

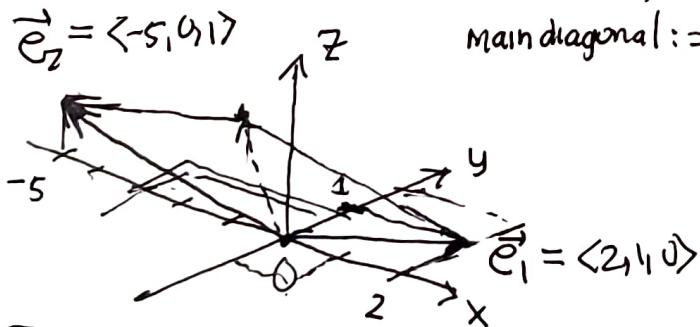
example Find a basis of the plane: $x - 2y + 5z = 0$

augmented matrix: $\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$ already reduced

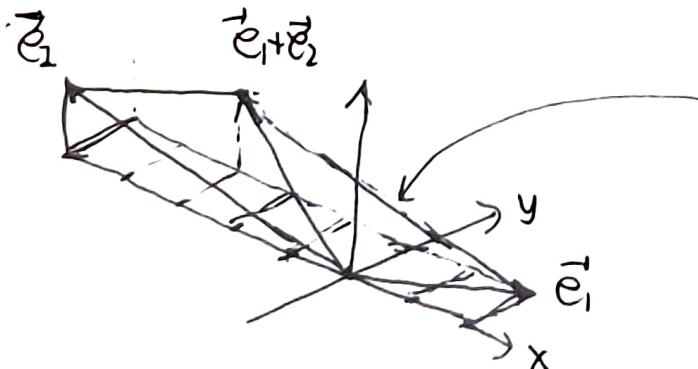
$$\left[\begin{array}{ccc|c} x & y & z & \\ \text{F} & \text{F} & \text{F} & \xrightarrow{\substack{x = t_1 \\ y = t_1 \\ z = t_2}} \end{array} \right] \rightarrow x = 2t_1 - 5t_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t_1 - 5t_2 \\ t_1 \\ t_2 \end{pmatrix} = t_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{e}_1 \quad \vec{e}_2 \quad \leftarrow \text{"coeff vectors"}$



Try again.



$$\vec{e}_1 + \vec{e}_2$$

main diagonal: $= \langle 2-5, 1+0, 0+1 \rangle = \langle \underbrace{-3}_{\text{above}}, \underbrace{1}_{\text{Second}}, \underbrace{1}_{\text{quadrant}} \rangle$

(tilts back)

on plane can tile with thrs
"unit coord" parallelogram to
make a coord grid.