

4.3

Span of a set of vectors, etc.

①

The textbook only spoke of linear independence of a set of vectors in  $\mathbb{R}^3$  in the first section but we have been emphasizing it for  $\mathbb{R}^n$  from the beginning

$m \times n$  linear system.  $A \vec{x} = \vec{0} \rightarrow$   $n$  cols  $\{\vec{v}_1, \dots, \vec{v}_n\} \in \mathbb{R}^m$

$$\underbrace{\langle v_1 | \dots | v_n \rangle}_{\text{columns}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \iff x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{0}$$

test for linear independence

$$\vec{x} = \langle x_1, \dots, x_n \rangle \quad \begin{array}{l} \text{coefficient vector} \\ \in \mathbb{R}^n \end{array}$$

- If  $\vec{x} = \vec{0}$  is the only soln, this is a linearly independent set
- If  $\vec{x} \neq \vec{0}$  it is a linearly dependent set and any vector whose coefficient is nonzero can be expressed in terms of the other vectors

MORE:

Classify the columns as (L) leading or (F) free according to the rref matrix.

- If no free columns, then  $\vec{x} = \vec{0}$  is the unique soln and the vectors are linearly independent
- For each free column, we get a free parameter in the solution

If there are  $p$  free columns ( $p$ -free variables):

$$\vec{x} = t_1 \vec{e}_1 + \dots + t_p \vec{e}_p \quad \left( \begin{array}{l} = \vec{0} \text{ implies } t_1 = \dots = t_p = 0 \\ \text{so } \{\vec{e}_1, \dots, \vec{e}_p\} \in \mathbb{R}^n \text{ are lin. ind.} \end{array} \right)$$

↑  
arbitrary coefficients

The general solution is an arbitrary linear combination of these  $p$  coefficient vectors.

We need a name for this.

4.3

Span of a set of vectors etc

(2)

Definition The set of all possible linear combinations of a set of vectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is called the span of this set.

Since the set is closed under linear combination, it is automatically a subspace.

Since: linear combinations of linear combinations are again linear combinations:

$$\begin{aligned} & a(c_1\vec{v}_1 + c_2\vec{v}_2) \\ & + b(c_3\vec{v}_1 + c_4\vec{v}_2) \\ = & a\vec{c}_1\vec{v}_1 + a\vec{c}_2\vec{v}_2 \\ & + b\vec{c}_3\vec{v}_1 + b\vec{c}_4\vec{v}_2 \\ = & (ac_1 + bc_3)\vec{v}_1 + (ac_2 + bc_4)\vec{v}_2 \end{aligned}$$

The vectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$  are said to "span" the subspace.

Returning to  $Ax = 0 \rightarrow \vec{x} = t_1\vec{e}_1 + \dots + t_p\vec{e}_p$  general soln

1) The solution space is the span of  $\{\vec{e}_1, \dots, \vec{e}_p\}$   
which are p-independent vectors. [p-plane thru origin of  $\mathbb{R}^n$ ]

2) The span of the cols of A  $\{\vec{v}_1, \dots, \vec{v}_n\}$   
is a subspace of  $\mathbb{R}^m$  spanned by these columns,  
but also by the subset of leading column vectors alone

If there are p free columns, there are  $n-p=r$  leading  
columns. r is the number of nonzero rows of the reduced  
matrix and is called the rank of the matrix.

[r-D-plane = r-plane thru origin]  
of  $\mathbb{R}^m$

### 4.3 Span of a set of vectors etc

(3)

Expressing vectors in  $\mathbb{R}^n$ :  $n$  standard basis vectors are understood

$$\mathbb{R}^2: \vec{x} = \langle x_1, x_2 \rangle = x_1 \langle 1, 0 \rangle + x_2 \langle 0, 1 \rangle = x_1 \hat{i} + x_2 \hat{j}$$

$$\mathbb{R}^3: \vec{x} = \langle x_1, x_2, x_3 \rangle = x_1 \langle 1, 0, 0 \rangle + x_2 \langle 0, 1, 0 \rangle + x_3 \langle 0, 0, 1 \rangle = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$$

$$\mathbb{R}^n: \vec{x} = \langle x_1, \dots, x_n \rangle = x_1 \langle 1, 0, \dots, 0 \rangle + \dots + x_n \langle 0, \dots, 0, 1 \rangle$$

We express all vectors as unique linear combinations of these  $n$  linearly independent vectors.

We build our rectangular coordinate systems with these "standard basis vectors!"

$\begin{cases} = 0 \\ \text{implies} \\ x_1 = \dots = x_n \\ = 0 \\ \text{i.e.} \\ \vec{x} = 0 \end{cases}$

We want to generalize them to any  $n$ -independent vectors in  $\mathbb{R}^n$ .

How to tell if  $n$  vectors in  $\mathbb{R}^n$  are linearly independent?

$$\langle v_1 | \dots | v_n \rangle \left[ \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right] = \emptyset \Leftrightarrow x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{0} \text{ has only the zero soln.}$$

$\begin{matrix} n \times n \\ \text{matrix} \\ A \end{matrix}$

$$\xrightarrow{\text{rref}} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]_{n \times n}$$

no free variables so can only reduce to the identity matrix so  $A$  has an inverse

$$Ax = \emptyset \rightarrow x = A^{-1}\emptyset = \emptyset$$

Therefore  $|A| \neq 0$ !

Determinant ( $\langle v_1 | \dots | v_n \rangle$ )  $\neq 0$  means  $\{\vec{v}_1, \dots, \vec{v}_n\}$  are linearly independent.