

3.3

row reduction solution of linear systems

①

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

mxn linear system

$$\rightarrow A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

matrix multiplication next

matrix form of linear system

$$C = [A \mid b]$$

augmented matrix

(x_1, \dots, x_n) free variables
 leading variables }

$$\downarrow rref(C')$$

Gauss-Jordan reduction

solve for leading variables in terms of the free variables
 which are set equal to parameters (if consistent)

RHS column matrix $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$\left\{ \begin{array}{l} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \text{OR} \\ \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \right.$

"homogeneous" linear system
 "nonhomogeneous" linear system

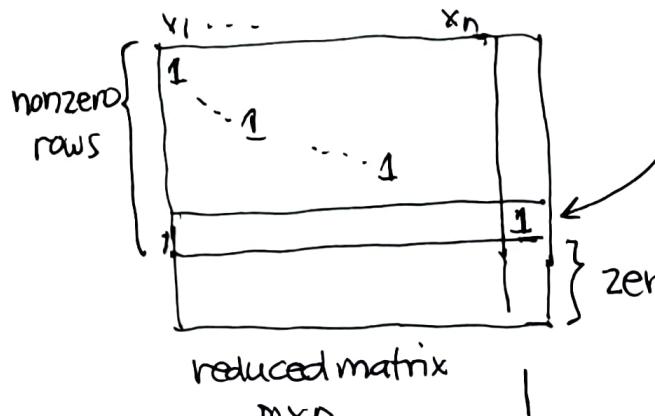
These two kinds of linear systems turn out to be very different, and have different interpretations.

This will be developed in the rest of Chapter 3 and in Chapter 4.

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row reduction solution of linear systems

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What leads to inconsistent systems?

if a leading 1 occurs in the last column
the corresponding equation is

$$0x_1 + \dots + 0x_n = 1$$

$0 = 1$ not possible!

existence of a soln to these equations
would imply $0=1$ so there cannot
exist a soln. **[INCONSISTENT SYSTEM]**

otherwise
consistent
system

if no leading 1 occurs in the last column
all the nonzero reduced eqns can be solved
for the leading variables
in terms of the free variables
 $\#$ free parameters
in soln.

no free variables means **unique soln**

Where do the zero rows come from?

example $\begin{cases} x+y+z=1 \\ -2x+2y+3z=0 \end{cases}$

add sum: $-x+3y+4z=1$

2×3 system

now 3×3 system but
the last equation is
redundant —
automatically satisfied
if first two are

$\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ -2 & 2 & 3 & 0 \\ -1 & 3 & 4 & 1 \end{array} \right] \xrightarrow{R_3 = R_1 + R_2} \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ -2 & 2 & 3 & 0 \\ 0 & 4 & 5 & 1 \end{array} \right]$

$\xrightarrow{\text{rref}} \left[\begin{array}{cccc} 1 & 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\begin{cases} x - \frac{1}{4}z = \frac{1}{2} \\ y + \frac{5}{4}z = \frac{1}{2} \\ 0 = 0 \end{cases}}$

of course can only solve
with only 2 independent
equations.

zero rows reflect redundancies in the system
we will quantify this in chapter 4.

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Solving Linear Systems Example

reduced augmented matrix for a system of 5 linear equations in 7 unknowns:

$$\left[\begin{array}{ccccccc|c} L & F & L & F & F & L & L \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & -2 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{matrix} \downarrow & \downarrow & \downarrow \\ t_1 & t_2 & t_3 \end{matrix}$$

$$\begin{aligned} (x_1) - 2x_2 + x_4 &= 2 \rightarrow x_1 = 2x_2 - x_4 + 2 \\ (x_3) + 3x_4 &= 3 \rightarrow x_3 = -3x_4 + 3 \\ (x_6) &= 0 \rightarrow x_6 = 0 \\ (x_7) &= 6 \rightarrow x_7 = 6 \\ 0 &= 0 \end{aligned} \quad \begin{matrix} \text{convenient equivalent} \\ \text{system} \end{matrix} \quad \begin{matrix} \text{convenient equivalent} \\ \text{system} \\ \text{NOT THE SOLUTION!} \end{matrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 + 2 \\ x_2 \\ -3x_4 + 3 \\ x_4 \\ x_5 \\ 0 \\ 6 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 2t_1 - t_2 + 2 \\ t_1 \\ -3t_2 + 3 \\ t_2 \\ t_3 \\ 0 \\ 6 \end{bmatrix}$$

This is the solution

all variables expressed
in terms of the free
variables which may
take arbitrary values.
(not the solution)
(this is the result of
MAPLE solve)

MAPLE
rref
backsub
(reverse
order
parameter
indexing)

what does this mean?

any linear system of equations whose augmented matrix rref's to the above matrix will have this solution, which can be checked by backsubstitution of the soln into those equations.

For example this system:

$$\begin{array}{ll} -x_1 + 2x_2 + 4x_3 + 11x_4 & -4x_6 + x_7 = 16 \\ x_1 - 2x_2 + x_3 + 4x_4 & -2x_6 = 5 \\ -9x_3 - 12x_4 & +2x_6 + 4x_7 = 12 \\ -3x_1 + 6x_2 - 4x_3 - 15x_4 & +2x_6 - 4x_7 = -42 \\ -4x_1 + 8x_2 - x_3 - 7x_4 & -x_6 + 3x_7 = 7 \end{array}$$

For simplicity we only check
the third equation in the system:

$$\begin{aligned} -4(-3t_2 + 3) &\stackrel{?}{=} 12 \\ -12(t_2) & \\ +2(0) & \\ +4(6) & \\ \hline = 12t_2 - 12 - 12t_2 + 24 & \\ = 12 & \checkmark \end{aligned}$$

only when you've checked all 5 equations can you
be sure it really is a solution.

Solving Linear Systems

$$A\vec{x} = \vec{b} \xleftarrow[\text{mxn nx1 mx1}]{\substack{\text{equivalent} \\ \text{systems} \\ (\text{same solns } \vec{x})}} \rightarrow \bar{A}\vec{x} = \bar{\vec{b}}$$

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↑ new equations

↓

augmented matrix $[A, \vec{b}]$ $\xrightarrow[\text{row reduce}]{\text{completely}} \text{rref}([A, \vec{b}]) = [\bar{A}, \bar{\vec{b}}]$

Let $r = \text{rank}(A)$

consistent case

independent equations in original system

$r \leq m$
nonzero rows

$$\left\{ \begin{array}{c|cccc|c} & x_1 & & & x_n & \\ \hline 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & & & & \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array} \right.$$

redundant equations in original system

$m-r$
zero rows

r leading 1's among n columns ($\text{so } r \leq n$)

inconsistent case

leading 1 in last column

$$[0 \dots 0 : 1]$$

corresponds to equation

$$0x_1 + \dots + 0x_n = 1$$

$$\text{or } 0=1$$

zero rows lead to trivial equations

$$0x_1 + \dots + 0x_n = 0$$

$$\text{or } 0=0$$

Each of the first n columns corresponds to one of the variables x_1, \dots, x_n . Label the leading entry column variables as bound, the remaining $n-r$ variables as free, or "leading"

For a given leading entry 1, in a given row, the corresponding equation can be solved for the corresponding bound variable (since it has a unit coefficient) in terms of the free variables. Since each bound variable only appears in one spot in the reduced system (zero's above and below its leading one coefficient), it does not enter into any other equation.

Set the free variables equal to arbitrary constants t_1, t_2, \dots, t_{n-r} and back-substitute these values into the equations solved for the bound variables, thus expressing all the variables in terms of these "parameters." The solution is a parametrized representation of the points in the solution space of the system.

terminology: different books use different names

free variables = independent variables

(freely specify \rightarrow parameters)

leading variables = bound variables = dependent variables

(solve for)

also, changing the order of the variables changes their classification

Ex. eqn for straight

line: $y = mx+b$
has 2 different solns
by rref technique
depending on order
(x,y) or (y,x)

$$mx - y = b \xrightarrow{\text{reduce}} x - \frac{y}{m} = \frac{b}{m} \xrightarrow{\substack{\text{back} \\ \text{sub}}} \begin{cases} x = b/m + t_1/m \\ y = t_1 \end{cases}$$

or

$$y - mx = b \xrightarrow{\text{(already reduced)}} \xrightarrow{\substack{\text{back} \\ \text{sub}}} \begin{cases} x = t_2 \\ y = b + mt_2 \end{cases}$$

\uparrow \uparrow

standard form

where $t_2 = b/m + t_1/m$
or $t_1 = b + mt_2$

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row reduction solution of linear systems

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how do we get a unique solution?

example $\begin{array}{l} x_1 = 1 \\ 3x_3 \\ \text{system} \\ (\text{reduced}) \end{array}$

$$\begin{array}{rcl} x_1 & = 1 \\ x_2 & = 4 \\ x_3 & = 3 \end{array}$$

main diagonal

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 0 \end{array} \right]$$

ones down the main diagonal
no "free columns" (zero columns before last)

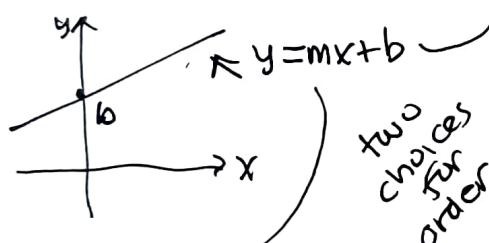
OR 4×3 system (reduced)

$$\begin{array}{rcl} x_1 & = 1 \\ x_2 & = 4 \\ x_3 & = 3 \\ 0 & = 0 \end{array} \leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \left\{ \begin{array}{l} \text{original system} \\ \text{had 1 redundant} \\ \text{equation} \end{array} \right.$$

why does the order of the variables matter

example:



$$-mx + y = b \quad \begin{array}{l} \text{standard form!} \\ \text{variables on LHS} \\ \text{constants on RHS} \end{array}$$

$$[-m \ 1 \ b]$$

↓ reduce

$$\left[1 \ -\frac{1}{m} \ \frac{b}{m} \right]$$

$$x - \frac{y}{m} = -\frac{b}{m} \rightarrow x = \frac{y-b}{m} = \frac{t_2-b}{m}$$

$$\left[\begin{array}{l} F \\ y=t_2 \end{array} \right]$$

$$\left[\begin{array}{l} x \\ y \end{array} \right] = \left[\begin{array}{l} (t_2-b)/m \\ t_2 \end{array} \right]$$

OR

$$y - mx = b$$

$$\left[1 \ -m \ b \right] \text{ reduced!}$$

solve for $y = mx + b = t_1 + b$

$$x = t_1$$

$$\left[\begin{array}{l} x \\ y \end{array} \right] = \left[\begin{array}{l} t_1 \\ t_1 + b \end{array} \right]$$

$$\begin{aligned} t_1 &= (t_2-b)/m && \text{linear} \\ t_2 &= t_1 + b && \text{change} \\ &&& \text{of} \\ &&& \text{parameter} \end{aligned}$$

but describe same line!

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nonstandard form!

$$2x - y + 1 = x + 3y - 4$$

← linear expressions
← on both sides

$$-x + y + 2 = 3x - y + 1$$

$$\text{LHS} = \text{RHS}$$

↓
"separate"
variables
and constants

$$\downarrow \quad \text{LHS} - \text{RHS} = 0$$

$$\text{OR} \quad \text{RHS} - \text{LHS} = 0$$

$$2x - x - y - 3y \neq -1 - 4$$

$$-x - 3x + y + y = 1 - 2$$

↓

$$x - 4y = -5$$

$$-4x + 2y = -1$$

we always assume
this preliminary
work has been done

homogeneous systems are always consistent

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

↑ last column of rref augmented matrix
always zero, system always consistent!

$$x_1 = x_2 = \dots = x_n = 0 \quad \text{"trivial soln"}$$

otherwise "nontrivial"

free variables = # free parameters in soln

no free variables, the zero soln is unique (only soln!)

Balancing chemical reactions

= solving homogeneous system

Friday