

1.5c linear 1st order DEs: mixing tank problems

(1)

"Mixing tank" problems are described by linear 1st order DEs.

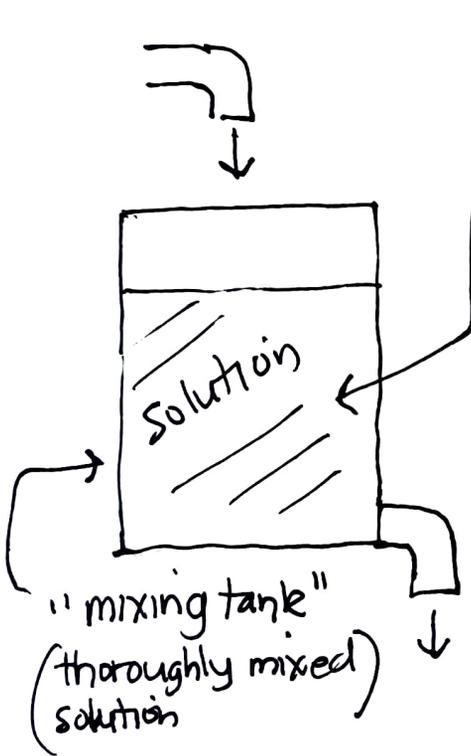
single mixing tank

(later we do multiple tanks)

useful examples - you don't have to be a ChemE student to have ~~any~~ intuition about how these behave

Setup:

input flow: { rate r_i (constants) input solution
concentration c_i



$X(t)$ amount of solute in solution in tank
 $V(t)$ volume of solution in tank
 $C_0(t) = \frac{X(t)}{V(t)}$ concentration of solute in tank

output flow: { rate r_o (constant)
concentration $C_o(t)$

$t=0$ start initial conditions: $X(0) = X_0$

$V(0) = V_0$

$V(t) = V_0 + \underbrace{(r_i - r_o)}_{\text{net rate of flow out = in - out}} t$ (linear function!)

(= V_0 if $r_o = r_i$)

1.5c Linear 1st order DEs: mixing tank problems

units tell us how to combine quantities.

$x(t) \sim$ mass or weight : kg or lb etc
 $V(t) \sim$ liquid volume : L or gal etc
 $c(t) \sim \frac{x(t)}{V(t)} \sim \frac{\text{mass or wt}}{\text{liquid vol.}}$ $\frac{\text{kg}}{\text{L}}$ or $\frac{\text{lb}}{\text{gal}}$ etc
 $r \sim \frac{\text{liquid vol.}}{\text{time}}$ $\frac{\text{L}}{\text{s}}$ or $\frac{\text{gal}}{\text{hr}}$ etc.
 $rc \sim \frac{(\text{mass or wt})}{\text{time}}$ **conals!**
 $rc \Delta t \sim \Delta(\text{mass or wt})$

$$\frac{dV(t)}{dt} = r_i - r_o$$

$$V(t) = V_0 + (r_i - r_o)t$$

DE derivation During time interval Δt :

$$\Delta x = (\text{amount in}) - (\text{amount out})$$

$$\approx (r_i c_i) \Delta t - (r_o c_o(t)) \Delta t$$

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o(t)$$

$$\frac{dx}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = r_i c_i - r_o c_o(t)$$

$$\frac{x(t)}{V(t)} = \frac{x(t)}{V_0 + (r_i - r_o)t}$$

$$\frac{dx}{dt} + \frac{r_o}{V_0 + (r_i - r_o)t} x = r_i c_i, \quad x(0) = x_0 \quad \text{IVP}$$

if $r_o = r_i \rightarrow \frac{r_o}{V_0}$ constant \rightarrow integral linear in t
 $r_o \neq r_i \rightarrow$ logarithmic integral in t

plug in parameters, solve DE.

1.5c linear 1st order DES: mixing tank problems

(3)

example: Great Lakes pollution → tank = Lake Huron!



cleaner water enters,
lowers pollutant
concentration

$$r_i = r_o = r = 350 \frac{\text{km}^3}{\text{yr}}$$

$$V = V_o = 480 \text{ km}^3$$

Goal: Answer question "How long does it take to reduce the "tank" concentration from 5 times the incoming concentration to twice that value?"

$$\frac{x_o}{V_o} = C_o(0) = 5c_i \quad \text{so } x_o = 5c_i V_o$$

(c_i left as
unknown parameter)

$$\text{DE: } \left[\frac{dx}{dt} + \frac{r}{V_o} x = r c_i \right] \rightarrow \frac{d}{dt} \left(e^{\frac{r}{V_o} t} x \right) = r c_i e^{\frac{r}{V_o} t}$$

$$e^{\int \frac{r}{V_o} dt} = e^{\frac{r}{V_o} t}$$

$$x e^{\frac{r}{V_o} t} = \int r c_i e^{\frac{r}{V_o} t} dt = \frac{r c_i}{r/V_o} e^{\frac{r}{V_o} t} + \mathcal{C}$$

$$x = e^{-rt/V_o} (c_i V_o e^{rt/V_o} + \mathcal{C})$$

difference decays
to equilibrium soln

$$\Rightarrow x = c_i V_o + \mathcal{C} e^{-rt/V_o} \quad \text{gen soln}$$

$$5c_i V_o = x_o = c_i V_o + \mathcal{C} \rightarrow \mathcal{C} = 4c_i V_o$$

$$x = c_i V_o + 4c_i V_o e^{-rt/V_o} = c_i V_o (1 + 4e^{-rt/V_o})$$

When does $x(t) = 2c_i V_o$?

$$2 = (1 + 4e^{-rt/V_o}) \rightarrow e^{-rt/V_o} = \frac{2-1}{4} = \frac{1}{4} \rightarrow e^{rt/V_o} = 4$$

$$\frac{rt}{V_o} = \ln 4 \rightarrow t = (\ln 4) \frac{V_o}{r}$$

τ ! sets scale of answer! = time it takes for V_o to flow thru lake

1.5c linear 1st order DEs: mixing tank problems

④

Put in the numbers

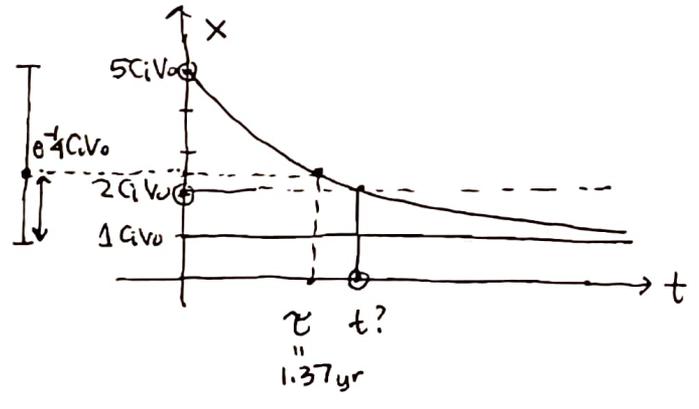
$$t = \left(\frac{V_0}{r} \right) \ln 4 = \left(\frac{480 \text{ km}^3}{350 \text{ km}^3/\text{yr}} \right) (1.3862) = 1.9012$$

$$\approx 1.90 \text{ yrs}$$

$\tau = 1.3714 \text{ yr}$

pictures are valuable!
 hand diagram captures big picture
 (technology makes it precise)
 typical decaying exponential behavior!

E&P3 1.5.37 Great Lakes Pollution: the diagram & τ



exponential decay of difference $\sim e^{-\frac{r}{V_0}t}$

$$\tau = \frac{V_0}{r} = \frac{480 \text{ km}^3}{350 \text{ km}^3/\text{yr}} = 1.37 \text{ yr}$$

(time to fill empty lake at this flow rate)

We found $t = 1.9 \text{ yr}$, a bit longer than τ as the diagram suggests.

If we waited till the difference was 1% of the initial difference $4C_iV_0$, it would be $t = 4.6 \tau = 6.3 \text{ yr}$.

Knowing the difference decays exponentially, we could have answered the final question immediately: how long does the level reduce from 5 to 2 times the asymptotic value of $1C_iV_0$?

$$4e^{-t/\tau} = 1 \rightarrow -\frac{t}{\tau} = \ln \frac{1}{4} \rightarrow t = \tau \ln 4 = (1.37)(1.386) = 1.92 \approx 1.9 \text{ yr}$$

$(t = \tau(-\ln \frac{1}{4})) =$

$(e^{-t/\tau} = \frac{1}{4})$
(take ln both sides)

initial difference factor = 5-1
 final difference factor = 2-1

1.5c linear 1st order DEs: mixing tank problems

(5)

$$V(t) = V_0 + (r_i - r_0)t \begin{cases} r_0 > r_i & \text{volume decreases to zero (stop)} \\ r_0 < r_i & \text{volume increases to max volume (stop)} \end{cases}$$

integrating factor:

$$\int P(t) dt = \int \frac{r_0 dt}{V_0 + (r_i - r_0)t} = \frac{r_0}{r_i - r_0} \int \frac{du}{u} = \frac{r_0}{r_i - r_0} \ln V(t) = \ln V(t)^{\frac{r_0}{r_i - r_0}}$$

$$e^{\int P(t) dt} = e^{\ln V(t)^{\frac{r_0}{r_i - r_0}}} = V(t)^{\frac{r_0}{r_i - r_0}}$$

exponent $\frac{r_0}{r_i - r_0} < 0$ if $r_0 > r_i$ decreasing vol
 > 0 if $r_0 < r_i$ increasing vol

We could write down the general solutions easily done by hand up until the RHS integral but better left to a Maple worksheet.

Gives understanding of how parameter values affect the variable $x(t)$.