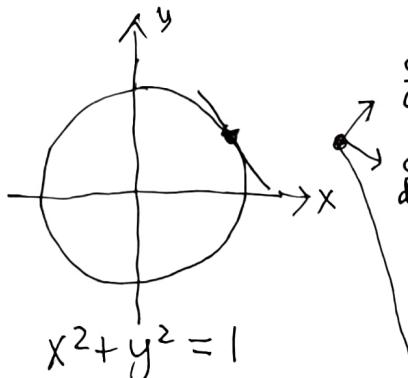


1.5b

linear 1st order DEs: complications

①

switching independent & dependent variables: $x \leftrightarrow y$ 

we can choose which is which here!

$$\frac{d}{dx} [x^2 + y^2 = 1] \rightarrow 2x + 2y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d}{dy} [x^2 + y^2 = 1] \rightarrow 2x \frac{dx}{dy} + 2y = 0 \rightarrow \frac{dx}{dy} = -\frac{y}{x}$$

reciprocals!

$$\boxed{\frac{dx}{dy} = \frac{1}{dy/dx}}$$

Now consider this DE:

$$\frac{dy}{dx} = \frac{1}{y-2x} \xrightarrow{\text{"}\leftrightarrow\text{y"\}} \frac{dx}{dy} = y-2x \quad \begin{array}{l} \text{linear in } x \text{ as unknown-} \\ \text{put in standard form} \end{array}$$

$$e^{2y} \left[\frac{dx}{dy} + 2x = y \right] \rightarrow \frac{d}{dy} (x e^{2y}) = y e^{2y}$$

$\int 2dx = e^{2y}$

$$x e^{2y} = \int y e^{2y} dy$$

$$= \frac{1}{4} (2y-1) e^{2y} + C \quad \text{Maple!}$$

$$x = e^{-2y} \left[\frac{1}{4} (2y-1) e^{2y} + C \right]$$

$\cdot x = \frac{1}{4} (2y-1) + C e^{-2y}$

implicit:
we can't invert!

gen soln

$$\text{initial condition } y(0)=0 \Leftrightarrow x=0, y=0: \quad 0 = \frac{1}{4}(-1) + C(1) \rightarrow C = \frac{1}{4}$$

$$\boxed{x = \frac{1}{4} (2y-1 + e^{-2y})}$$

explicit!

Maple SolveDE interactively, symbolically,
explicit: change to NO for implicit soln

$$y(x) = \frac{1}{2} \underbrace{\text{LambertW}(-4 \cdot C_1 e^{-4x-1})}_{\text{special function: solve}(x = y e^y, y)} + 2x + \frac{1}{2}$$

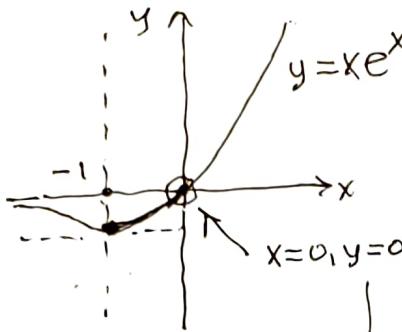
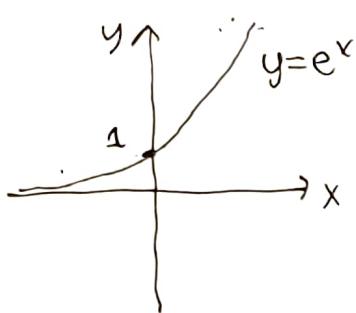
special function: $\text{solve}(x = y e^y, y) = \text{LambertW}(x)$

1.5b

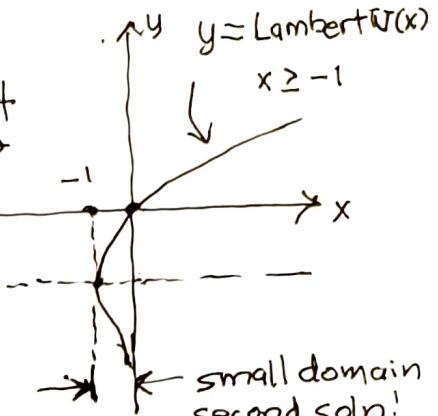
Linear 1st order DEs: complications

②

What is this special function? Not mysterious!



invert



$$\frac{d}{dy} [X = ye^y] \Leftrightarrow y = \text{LambertW}(x)$$

$$\hookrightarrow \frac{dx}{dy} = (1+y)e^y$$

$$\frac{dy}{dx} = \frac{e^{-y}}{1+y}, \quad y(0) = 0$$

$$\text{Maple: } y(x) = \text{LambertW}(x)$$

DEs lead to the definition of many special functions.

1.5b) linear 1st order DEs: complications

(3)

definite integrals and the IVP solution

$$\text{IVP: } \frac{dy}{dx} = f(x), \quad y(a) = y_0.$$

↓ gen soln

$$y = \underbrace{\int f(x) dx}_{F(x)} + C \quad (\text{making additive constant explicit})$$

any particular antiderivative $F'(x) = f(x)$

$$\text{initial condition: } y_0 = y(a) = F(a) + C \rightarrow C = y_0 - F(a)$$

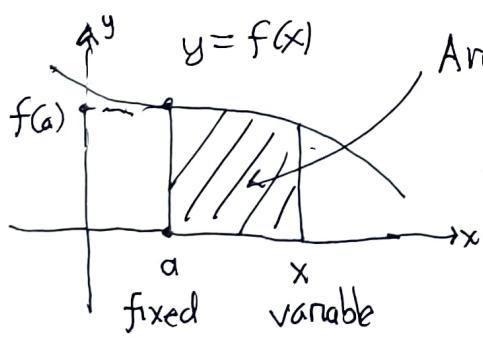
$$y = \underbrace{F(x) - F(a)}_{\downarrow} + y_0$$

$$y = \underbrace{\int_a^x f(t) dt}_{\downarrow} + y_0$$

this defines the particular antiderivative such that it vanishes at $x=a$.

If $f(x)$ does not have a simple antiderivative, we can simply define one using this definite integral expression! We just need to choose a particular reference point $x=a$ that is useful.

Such functions are called "accumulation functions"



$\text{Area}(x) = \int_a^x f(t) dt$ "signed" area increases from zero at a if $f(a) > 0$ as x increases from a : area accumulates under the curve!

[Watch Khan Academy short video]

1.5b

linear 1st order DEs : complications

(4)

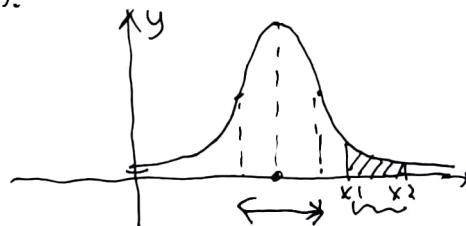
familiar example:

$$\frac{dy}{dx} = \frac{1}{x}, \quad x > 0$$

$$y = \int \frac{1}{x} dx = \underbrace{\int_1^* \frac{1}{t} dt + C}_{\ln|t| \Big|_1^x = \ln|x| - \ln 1} = \ln|x| \\ = \ln x, \quad x > 0$$

$\ln x = \int_1^x \frac{1}{t} dt$ is an area accumulating function!

HW problem 1.5.29 explores the most important such accumulation function needed for evaluating probability with a normal "bell curve" distribution.



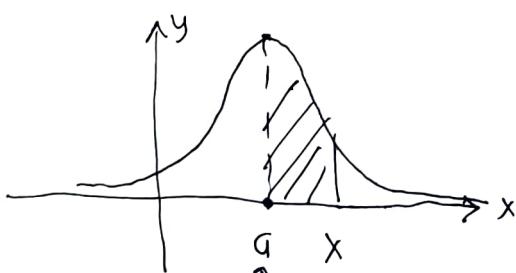
calc 2 topic 1 probability
= area under curve

area = probability that variable x assumes a value in the interval $x_1 \leq x \leq x_2$

$$P(x_1 \leq x \leq x_2) = F(x_2) - F(x_1)$$

area accumulation function called the cumulative distribution function

differences of areas = area in between



obvious reference pt $\rightarrow F(x) = \int_a^x f(t)dt$

BUT

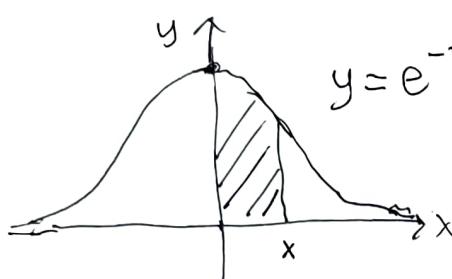
we are not doing probability this semester!

1.5b

linear 1st order DEs: complications

(5)

HW 1.5.29 example.



$$\frac{dy}{dx} = e^{-x^2}, \quad y(0) = 0$$

$$y = \int_0^x e^{-t^2} dt$$

↓ Maple

$$y(x) = \frac{\sqrt{\pi}}{2} \underbrace{\text{erf}(x)}_{\text{"error function"}}$$

↑ "error function"

constant needed for normal distributions

area accumulation
function

no expression
for antiderivative
expressible in
terms of familiar
functions!

direction fields for linear DEs:

previous examples:

$$\frac{dy}{dx} - y = e^{-x/3}, \quad y(0) = -1$$

$$\int -1 dx = e^{-x}$$

$$y = \underbrace{-\frac{3}{4}e^{-x/3}}_{\text{decaying exponential (antiderivative term)}} + C e^x \rightarrow y = -\frac{3}{4}e^{-x/3} - \frac{1}{4}e^x$$

decaying
exponential
(antiderivative)
term

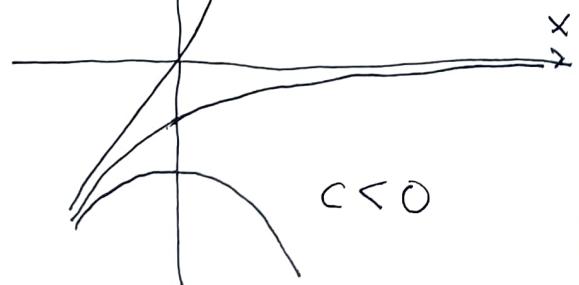
decays away:
 $\tau = 3$
 $5\tau \approx 15$

growing
exponential $\tau = 1$

$C > 0$	goes up	as x increases
$C = 0$	goes to zero	
$C < 0$	goes down	



$C > 0$



$C < 0$

$C = 0$ divides solutions into
2 subfamilies with very
different behavior

[typical of linear DE solns!]