

1.5a Linear 1st order DEs

①

Both sides of a linear DE are linear functions of the unknown and its derivatives.

first order case: linear in y , $\frac{dy}{dx}$ means $A \frac{dy}{dx} + By + C$
can depend on x ———— ↑ ↑ ↑

Standard form: put unknown terms on LHS, remaining on RHS, make derivative coefficient 1

$$\frac{dy}{dx} + P(x)y = Q(x)$$

↑ unit coefficient!

$P(x)$ is key to the solution algorithm

3 examples: $\begin{cases} P(x) \propto x^n, n \neq -1, 0 \\ P(x) = \text{const.} \\ P(x) \propto 1/x = x^{-1} \end{cases}$ ($x^0 = \text{const}$)

The solution algorithm involves preliminary algebra steps using the product rule and integration to reach a point where we can integrate both sides of the DE (as in the separable case).

PopTalk

This algorithm together with the separable DE algorithm give us a peek at the many special kinds of DEs that can be solved by some special recipe.

It also allows us to solve and understand how DEs useful for so many practical applications work.

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②

Example 1 $\frac{dy}{dx} (-2x)y = 3x \leftarrow$ (once in standard form)

① Integrate this coefficient

$$\int P(x) dx = \int -2x dx = -x^2 + \cancel{C}$$

pick any antiderivative \int not needed

② Exponentiate the result:

$$e^{\int P(x) dx} = e^{-x^2}$$

"integrating factor" $\left(\cancel{e^C} \right)$

multiplies both sides - cancels out!

③ multiply both sides of DE by this factor

$$e^{-x^2} \left(\frac{dy}{dx} - 2xy \right) = e^{-x^2} (3x)$$

④ Rewrite LHS as product derivative

$$\frac{d}{dx} (ye^{-x^2})$$

unknown times integrating factor

why? $= \frac{dy}{dx} e^{-x^2} + y e^{-x^2} (-2x)$
 $= e^{-x^2} \left(\frac{dy}{dx} - 2xy \right) !!$

Just the product rule!

⑤ $\int \frac{d}{dx} (ye^{-x^2}) dx = \int 3xe^{-x^2} dx$ integrate both sides

$$ye^{-x^2} = -\frac{3}{2} e^{-x^2} + C$$

(only need constant on RHS)

⑥ solve for y: $y = e^{x^2} \left[-\frac{3}{2} e^{-x^2} + C \right] = -\frac{3}{2} + Ce^{x^2}$

Done!

$$y = -\frac{3}{2} + Ce^{x^2}$$

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example 2.

$$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = 2$$

$\int \frac{3}{x} dx = 3 \ln|x|$
 $e^{3 \ln|x|} = (e^{\ln|x|})^3 = |x|^3 = \underline{x^3}$
 unnecessary - multiplies both sides

$$x^3 \left[\frac{dy}{dx} + \frac{3}{x}y = 2 \right]$$

$$x^3 \left(\frac{dy}{dx} + \frac{3}{x}y \right) = 2x^3$$

$$\frac{d}{dx}(yx^3)$$

this is the key step we have to remember

check!
 $\left[= \frac{dy}{dx}x^3 + y(3x^2) = x^3 \left(\frac{dy}{dx} + \frac{3}{x}y \right) \right]$

$$\frac{d}{dx}(yx^3) = 2x^3$$

← yx^3 is an antiderivative of $2x^3$

integrate: $yx^3 = \int 2x^3 dx$
 $= \frac{2x^4}{4} + C = \frac{x^4}{2} + C$

solve for y:

$$y = x^{-3} \left(\frac{x^4}{2} + C \right) = \frac{1}{2}x^{4-3} + Cx^{-3} = \frac{1}{2}x + \frac{C}{x^3}$$

$$y = \frac{1}{2}x + \frac{C}{x^3}$$

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example 3. $\frac{dy}{dx} = y + e^{-x/3}$

$e^{-x} \left[\frac{dy}{dx} - y = e^{-x/3} \right]$ (standard form)
 $\int -1 dx = -x$
 e

$$e^{-x} \left(\frac{dy}{dx} - y \right) = e^{-x} e^{-x/3}$$

Ⓢ $\frac{d}{dx} (y e^{-x}) = e^{-(1+1/3)x} = e^{-4x/3}$

$$y e^{-x} = \int e^{-4x/3} dx = \frac{e^{-4x/3}}{-4/3} + C = -\frac{3}{4} e^{-4x/3} + C$$

$y = e^x \left(-\frac{3}{4} e^{-4x/3} + C \right)$
 $= -\frac{3}{4} e^x e^{-4x/3} + C e^x$
 $= -\frac{3}{4} e^{-x/3} + C e^x$ done

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In general:

$$e^{\int P(x) dx} \left(\frac{dy}{dx} + \underset{\int P(x) dx}{P(x)} y = Q(x) \right) \rightarrow \frac{d}{dx} \left(y e^{\int P(x) dx} \right) = Q(x) e^{\int P(x) dx}$$
$$y e^{\int P(x) dx} = \underbrace{\int Q(x) e^{\int P(x) dx} dx}_{\text{pick any antiderivative}} + C$$

additive constant explicit

$$y = e^{-\int P(x) dx} \left(\int Q(x) e^{\int P(x) dx} dx + C \right)$$

$$y = e^{-\int P(x) dx} \int Q(x) e^{\int P(x) dx} dx + C e^{-\int P(x) dx}$$

any antiderivative

↑
explicit arbitrary constant

$$\text{If } Q(x) = 0 \text{ then } y = C e^{-\int P(x) dx}$$

Integral formula shows structure of final result
BUT not to be plugged into!

Use the step by step algorithm.