

1.4b

Separable DEs: exponential behavior

(-1)

Exponential growth & decay: characteristic constants

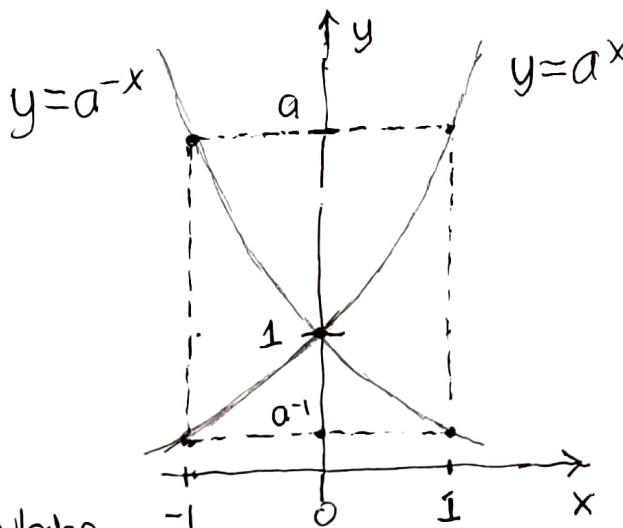
The "scale" of an object is a positive constant that characterizes how big something is: the radius  $r$  or diameter  $D$  of a circle, the side  $s$  of a square, the period  $T$  (time) or wavelength  $\lambda$  (length) of a sine/cosine oscillation.

For exponential growth/decay this ["characteristic scale"] or characteristic constant characterizes the interval over which a quantity increases/decreases (scales up/scales down) by some fixed factor: 2 or  $e \approx 2.718$ .

without calc knowledge, doubling or halving by factors of 2 are the simplest ways of quantifying exponential behavior but with calc knowledge, increasing/decreasing by factors of  $e$  are natural.

$$\begin{array}{ll} \text{growth} & \text{decay} \\ a^x & a^{-x} = \frac{1}{a^x} \\ \downarrow & \downarrow \text{base } a > 1 \end{array}$$

When  $x \rightarrow x+1$  then  $a^x \rightarrow a^{x+1} = a a^x$  increases by factor of  $a$   
 $a^{-x} \rightarrow a^{-(x+1)} = a^{-1} a^{-x}$  decreases



With calc knowledge:

$$\frac{d}{dx} a^x = \frac{d}{dx} (e^{(\ln a)x}) = (\ln a) e^{(\ln a)x} = \underbrace{(\ln a)}_{\sim} a^x = 1 \text{ when } a = e$$

For every value of  $x$ , increasing its value by 1 changes  $a^{\pm x}$  by a factor of  $a^{\pm 1}$ .

Without calc knowledge people say: "What is  $e$ ?"

Euler's number is preferred for calculus!

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Separable DEs : exponential behavior

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Exponentials in applications

To use exponentials in applications we have to "scale" the variables

$$\begin{aligned} y &= y_0 e^{\pm kx} \\ &= y_0 e^{\pm \frac{x}{\tau}} \end{aligned} \quad \left. \begin{aligned} &= y_0 2^{\pm Kx} \\ &= y_0 2^{\pm \frac{x}{\tau}} \end{aligned} \right\}$$

but we have a choice of two bases but exponents must be dimensionless

$y(0) = \text{initial value}$

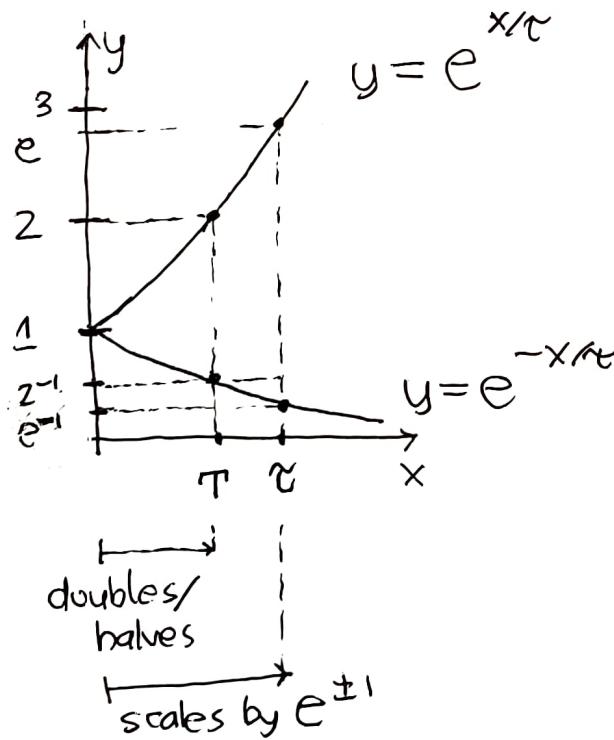
$$\begin{aligned} \pm \frac{1}{y} \frac{dy}{dx} &= k > 0 \quad \text{exp growth/decay "rate"} \\ &= \frac{1}{\tau} > 0 \rightarrow \tau = \text{characteristic "scale" "length" "time"} \end{aligned}$$

same units as  $x$  so

$\frac{x}{\tau}$  is dimensionless

Equate:  $2^{\frac{x}{\tau}} = e^{\frac{x}{\tau}}$

Take ln:  $\frac{x}{\tau} \ln 2 = \frac{x}{\tau} \rightarrow T = \underbrace{(\ln 2)}_{\approx 0.69} \tau \quad \text{or} \quad \tau = \frac{1}{(\ln 2)} T \approx 1.44$



We use the exponential function in all our calculations but translate to base 2 occasionally using:

$$2^x = (e^{\ln 2})^x = e^{(\ln 2)x}$$

just slightly different ways of characterizing exponential behavior.

### 1.4b Separable DEs: exponential behavior

$$\frac{dy}{dx} = Ry \leftarrow y = y_0 e^{kx}$$

↑  
exponential rate factor

initial value  
at  $x=0$

$$kx$$

$k > 0$  growth

$k < 0$  decay

(population)

(radioactivity)

$kx$  must be dimensionless to give value ind. of units, so  $k$  has units of  $\text{Y}/\text{x}$

or  $|\sqrt{k}|$  has units of  $\text{x}$

$$\tau \equiv 1/|k| > 0 \quad \text{characteristic scale}$$

(length/time)

$$y = y_0 e^{\pm \frac{x}{\tau}}$$

$$\frac{y}{y_0} = e^{\pm \frac{x}{\tau}}$$

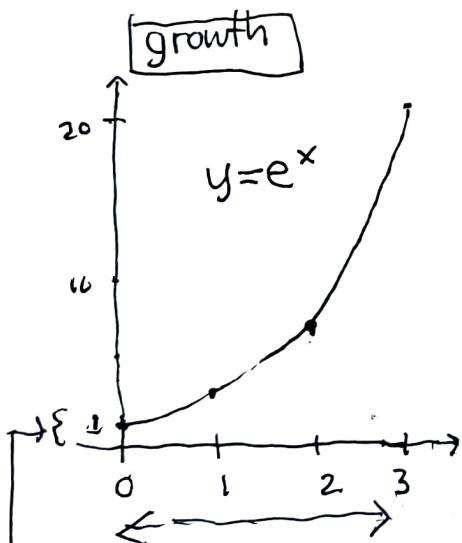
$$T = e^{\pm \frac{x}{\tau}}$$

$\frac{x}{\tau}$  dimensionless ratio

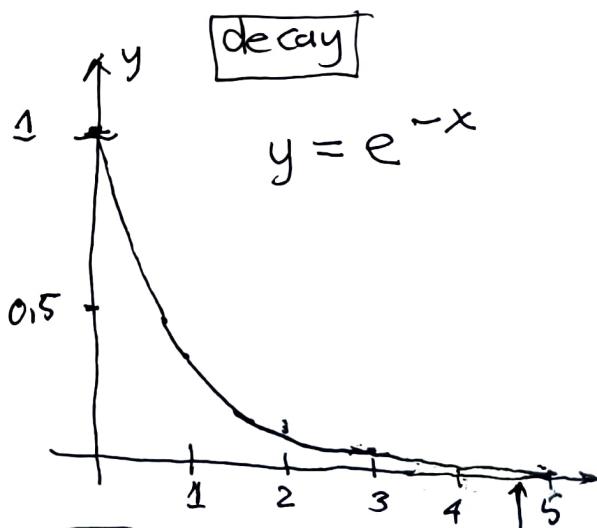
dimensionless ratios!

dimensionless variables

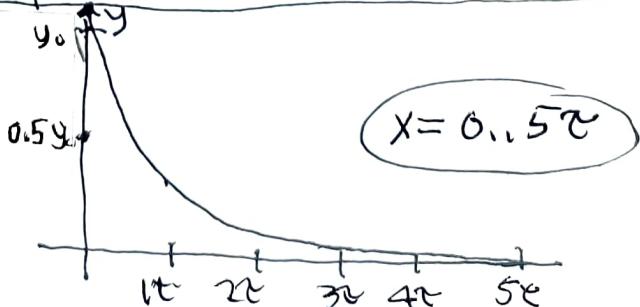
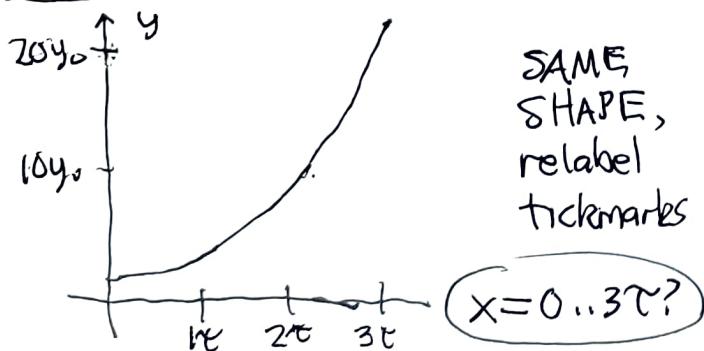
unit tickmarks appropriate for graphing



{ too see initial value,  
horizontal window }  
 $(x=0..3)$  at most



$x=0..5$   
shows approach  
to within 1%  
of initial value



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### 1.4b Separable DEs: exponential behavior

plotting an exponential function requires choosing the horizontal window with small integer multiples of the scale  $\tau$

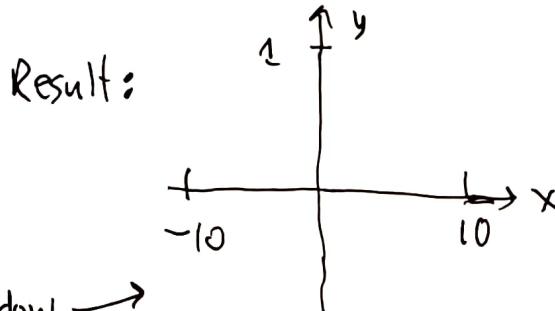
#### Example

Quantum penetration of the wavefunction of an electron into the surface of a conducting metal is described by a characteristic length of  $\tau = 10 \text{ nanometers}$   
 $= (10 \cdot 10^{-9} \text{ meters})$

$$\text{so } \tau = 10^{-8} \text{ meters}$$

$$1/\tau = 10^8$$

$$Q = e^{-\frac{x}{10^{-8}}} = e^{-10^8 x} \quad \text{plot this!}$$



usual window  $\rightarrow$

if not chosen otherwise

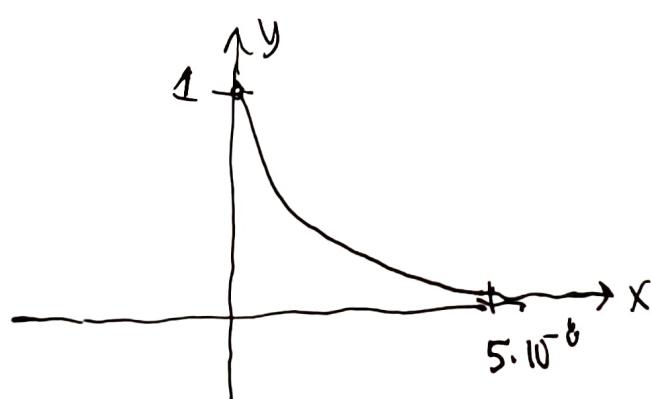
(graphing calculators too!)

NOTHING!

on this scale the graph is indistinguishable from the y-axis

See Maple

↓ choose window  
 $x = 0..5 \cdot 10^{-8}$



characteristic scale

$\tau$

Key to understanding exponential growth/decay

Return after Word Problem

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Separable DEs: exponential behavior

2b

Newton's law of cooling example:  $\frac{dT}{dt} = -k(T-A)$ ,  $T(0) = T_0$

A 1-lb roast initially at  $50^{\circ}\text{F}$   
is placed in a  $375^{\circ}\text{F}$  oven  
at 5pm.

After 75 minutes  
it is found that the temperature  $T(t)$   
of the roast is  $125^{\circ}\text{F}$ .

When will the roast be  $150^{\circ}\text{F}$   
(medium rare)

$$T_0 = 50$$

$$A = 375$$

$$t = 0$$

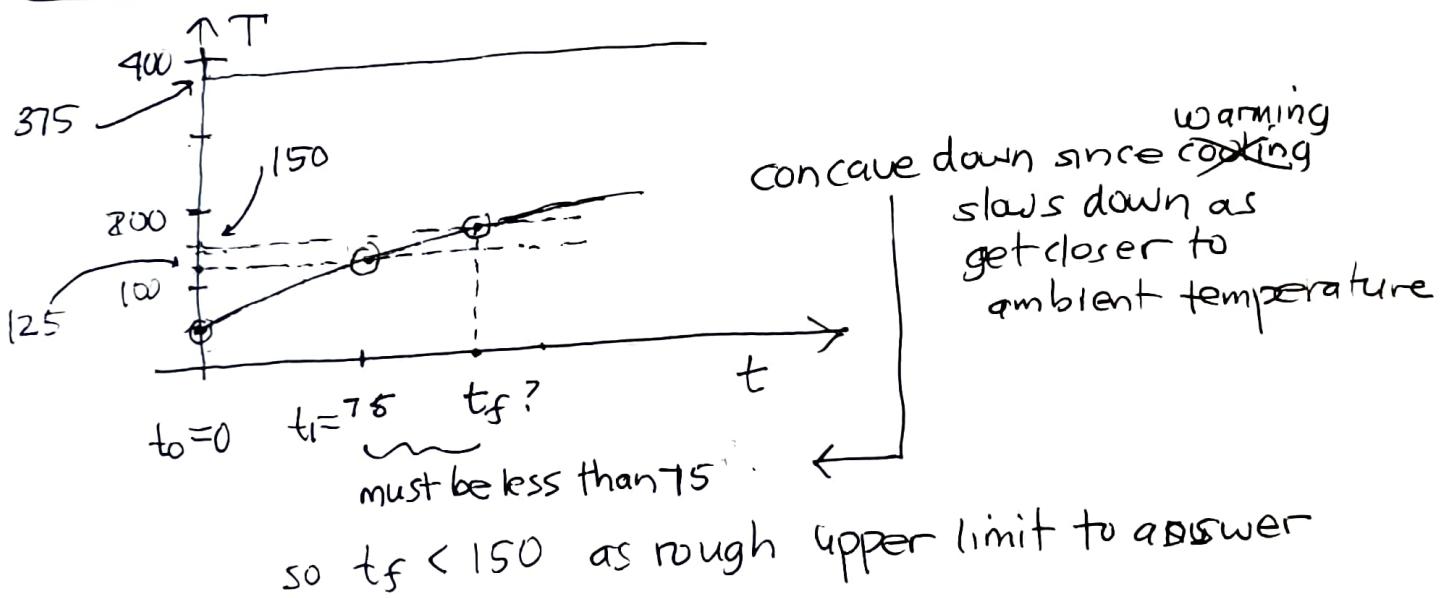
$$t_1 = 75$$

$$T(75) = 125$$

$$T(t') = 150 \ ?$$

$\uparrow$   
 $\text{? } t_f$

Draw a graph!



# 1.4b Separable DEs: exponential behavior

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Word Problem: Modeling cooking a roast (t minutes after) →

5pm

Newton's law  
of warming

$$\frac{dT}{dt} = -k(T-A), k > 0$$

$$\left[ \begin{array}{l} \text{exp. decay} \\ \frac{d}{dt}(T-A) = -k(T-A) \\ \text{difference decays} \end{array} \right]$$

unknown:  
need extra.  
condition

setup:

$$A = 375^\circ\text{F} \quad (\text{oven temp})$$

$$T(0) = 50^\circ\text{F} \quad (\text{roast goes in oven})$$

$$T(75) = 125^\circ\text{F} \quad (\text{second condition})$$

$$\boxed{T(t) = 150^\circ\text{F?}} \quad \text{when done}$$

goal: t in hours, minutes,  
clock time?

separable procedure

separate  $\frac{dT}{T-A} = -k dt \quad \leftarrow \text{division by } T-A \text{ ? cannot } = 0! \quad (T=A \text{ is equilibrium soln!})$

integrate  $\int \frac{dT}{T-A} = - \int k dt$

antider.  $\ln|T-A| = -kt + C_1$

exponentiate:  $\underbrace{e^{\ln|T-A|}}_{\text{rules of expts}} = e^{-kt+C_1} = e^{C_1} e^{-kt}$

simplify:  $|T-A| = e^{C_1} e^{-kt}$

$$\begin{aligned} T-A &= \underbrace{\pm e^{C_1}}_{\equiv C} e^{-kt} \quad \leftarrow \pm e^{C_1} \neq 0 \text{ ever!} \\ &\equiv C \quad \text{simpler constant allows } C=0 \\ &\quad \text{to recapture missing soln } T=A \end{aligned}$$

solve ↓  $T = A + Ce^{-kt}$

$$\boxed{T = 375 + C e^{-kt}} \quad \text{gen soln}$$

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Separable DEs : exponential behavior

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$$T = 375 + C e^{-kt}$$

gen soln

$$50 = T(0) = 375 + C \rightarrow C = 50 - 375 = -325$$

backsub

$$T = 375 - 325 e^{-kt}$$

IVP soln.  
need extra condition to determine

$$125 = T(75) = 375 - 325 e^{-75R}$$

solve for R

$$325 e^{-75R} = 375 - 125 = 250$$

$$\frac{325}{250} = e^{-75R} \quad \leftarrow \begin{cases} \text{minus eliminated} \\ \text{personal preference} \end{cases}$$

$$\frac{13}{10} = 1.3$$

↑  
exact triggers  
Maple decimal  
numbers

$$\ln\left(\frac{13}{10}\right) = \ln e^{-75R} = -75R \rightarrow R = \frac{\ln(13/10)}{75} \quad \text{backsub}$$

$$C \equiv 1/R = \frac{75}{\ln(1.3)} \approx 285.86 \quad (4.34 \text{ hours})$$

scale to raise roast to oven temp!  
but "cooked" much earlier.

$$T = 375 - 325 e^{-\frac{\ln(13/10)t}{75}}$$

"exact", but clearly decimal value relevant to problem

$$150 \stackrel{?}{=} T(t) = 375 - 325 e^{-kt}$$

$$-225 = -325 e^{-kt}$$

$$e^{kt} = \frac{325}{225} = \frac{13}{9}$$

$$Rt = \ln(13/9)$$

$$t = \underbrace{\ln(13/9)}_{0.37} \frac{1}{R} = \frac{\ln(13/9)}{\ln(13/10)} 75 \approx 105.119 \approx 60 + 45.1$$

1 hr, 45 minutes

about  $\frac{1}{3}$  char. time      5:00pm      6:45 pm

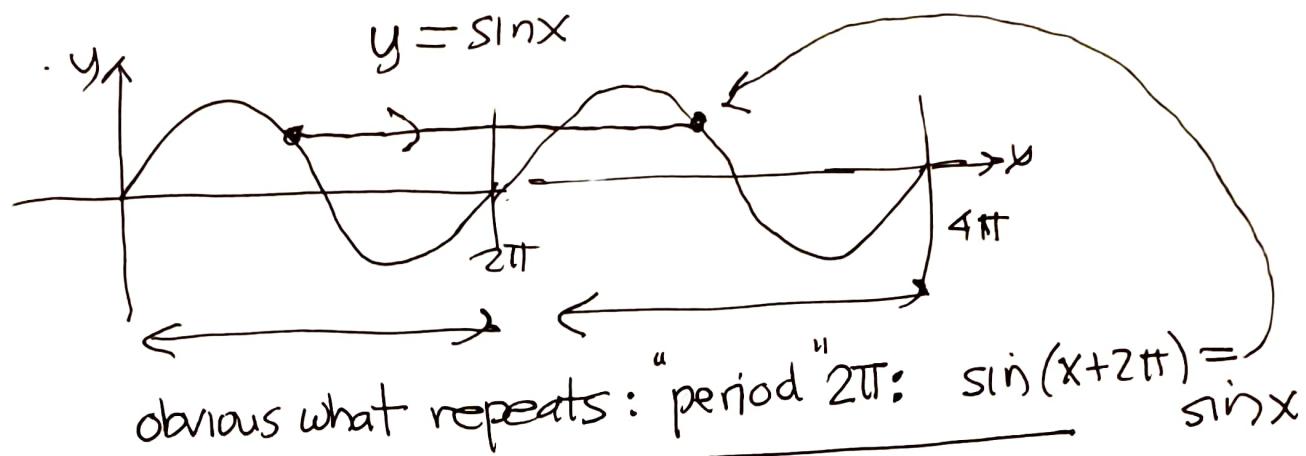
"The roast is done at 6:45pm"

See Maple plots

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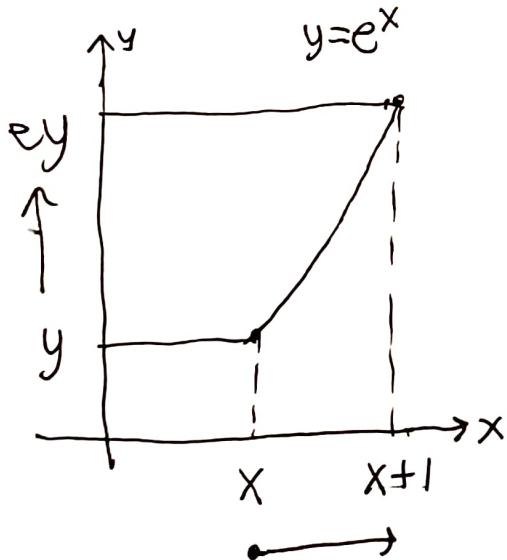
## Separable DEs: exponential behavior

5

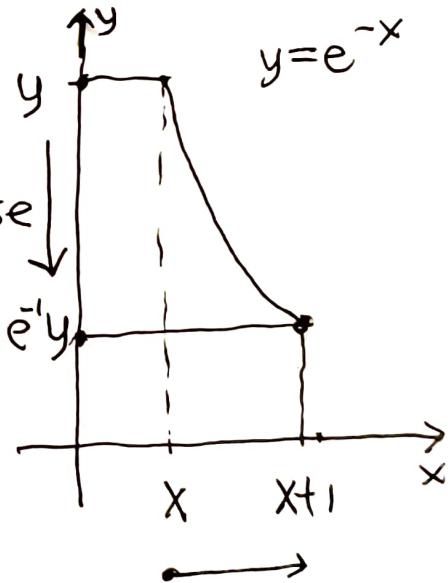


exponentials are more subtle:

$$\begin{aligned} e^{x+1} &= e^x e^1 = e^x e && \text{scaling property} \\ e^{-(x+1)} &= e^{-x} e^{-1} = e^{-x} e^{-1} && \text{repeats} \end{aligned}$$

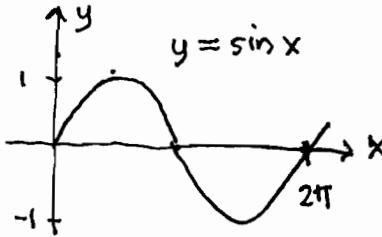


move over  
1 unit:  
increase/decrease  
value by a  
factor of  $e^y$

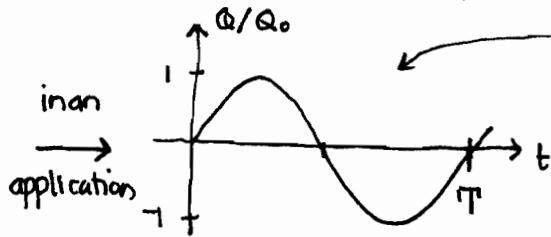


1 unit is this repetitive multiplicative  
"period"  $\rightsquigarrow$  characteristic interval

## characteristic constants for exponential behavior



periodic, each "cycle"  
repeats every interval  
of length  $2\pi$



$$Q = Q_0 \sin kt \quad \text{or}$$

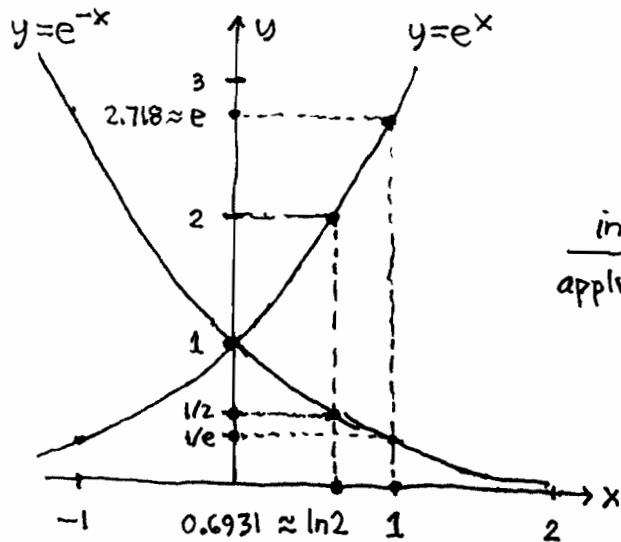
$$\frac{Q}{Q_0} = \sin kt \quad \text{if time}$$

"frequency"  $k > 0$   
= rate parameter  
units: inverse time

$$\sin(kt) = 2\pi \text{ at end of cycle} \rightarrow t = \frac{2\pi}{k} \rightarrow T = \frac{2\pi}{k}$$

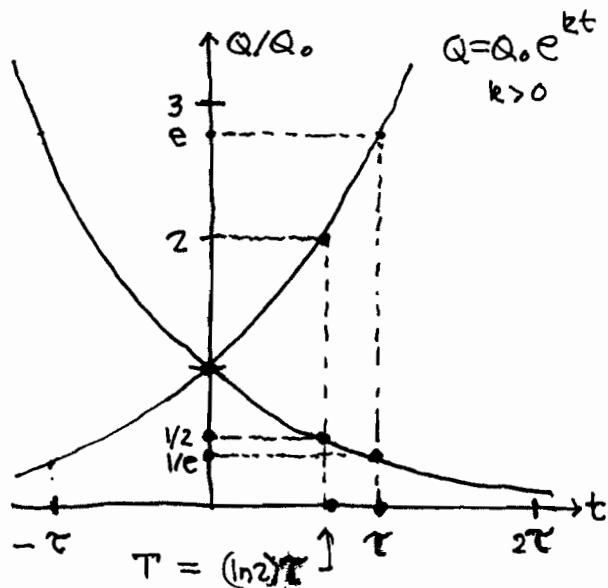
period: units = time  
sets scale for "viewing window":  
number of multiples of  $T$  in horizontal width (time interval)  
tells how many cycles you see

similarly:



$$Q = Q_0 e^{kt} \quad k < 0$$

in an application



$k$  is the rate factor (units: inverse time)

$$Q = Q_0 e^{kt} \quad \text{or} \quad \frac{Q}{Q_0} = e^{kt}$$

now length (or time) scale can be characterized by  
how long it takes to increase/decrease by a factor  
of  $e$  or 2 (two special numbers):

$$e \bullet \begin{cases} e^{kt} & k > 0 \\ e^{kt} & k < 0 \end{cases} \quad \begin{cases} e^{kt} = e \rightarrow kt = 1 \rightarrow t = 1/k \\ e^{kt} = e^{-1} \rightarrow kt = -1 \rightarrow t = -1/k = 1/|k| \end{cases} \quad \left\{ \begin{array}{l} \tau = \frac{1}{|k|} \\ (\text{"e-folding time"}) \end{array} \right. \quad \begin{array}{l} \text{"characteristic time (or length)" } \\ \text{units: time} \end{array} \quad \begin{array}{l} \text{"tau"} \\ \text{("tau")} \end{array}$$

$$2 \bullet \begin{cases} e^{kt} & k > 0 \\ e^{kt} & k < 0 \end{cases} \quad \begin{cases} e^{kt} = 2 \rightarrow kt = \ln 2 \rightarrow t = (\ln 2)/k \\ e^{kt} = 2^{-1} \rightarrow kt = \ln 2^{-1} = -\ln 2 \rightarrow t = -(\ln 2)/k = (\ln 2)/|k| \end{cases} \quad \left\{ \begin{array}{l} T = \frac{(\ln 2)}{|k|} \\ \text{k>0: "doubling time"} \\ \text{k<0: "half-life"} \end{array} \right. \quad \begin{array}{l} \text{("T ~ 70% T") } \\ \text{(T ~ 70% T)} \end{array}$$

Here what repeats every interval of length  $T$  (or  $\tau$ )  
is the doubling ( $k > 0$ ) / halving ( $k < 0$ ) of the value of  $Q$   
(or the increase / decrease by a factor of  $e$ ).

no need to remember formulas for characteristic times — can just "rederive" them  
when working a problem.