

1.4

Separable DEs

①

Solving DEs combines algebra with antiderivation (integration!) and indeed contains the latter as we have already seen as a special case. In the pre-technology days we had to use books listing thousands of types of functions that someone had figured out how to integrate over centuries. Similarly there are hundreds of kinds of DEs that have proved useful in mathematical and physical applications.

It is important to learn a few basic "recipes" for solving relatively simple DEs to take the mystery out of how they are found and because many very useful applications involve them.

For first-order DEs, we will study two recipes.

- 1.4 | separable DEs are only one step from antiderivation.
- 1.5 | linear DEs are a few steps away from antiderivation.
- 1.6 | exact DEs etc NO! not so useful, of marginal interest, touches on multivariable calc (not a prerequisite).

example : mixing algebra with diff/int.

$$\frac{d}{dx} \left(\frac{x-1}{x+1} \right) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

↑
"simplify"
(algebra)

easy to go back, quotient rule form obvious

now unrecognizable
how to go back?

↓ u-sub

$$\frac{x-1}{x+1} = \frac{x+1-1-1}{x+1} = 1 - \frac{2}{x+1}$$

$$\begin{aligned} & \int 2(x+1)^{-2} dx \\ &= \int 2u^{-2} du = 2 \frac{u^{-1}}{-1} + C \\ &= C - \frac{2}{x+1} \end{aligned}$$

? compare with $\frac{x-1}{x+1}$?

Solution recipes undo this mixing to reveal how to "go back" to the soln from the DE.

puzzling until algebra explains!

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Separable DEs

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$\frac{dy}{dx} = f(x, y) \leftarrow$ when this RHS function "separates" (factors!) into a product or quotient of the separate variables alone, we can "separate" the variables in the DE!

$$\frac{dy}{dx} = g(x) h(y) = \frac{g(x)}{h(y)}$$

where $h(y) = \frac{1}{j(y)}$

$\uparrow \quad \uparrow$ equivalent but right expression more convenient!

$$\left[\frac{dy}{dx} = \frac{g(x)}{j(y)} \right] \cdot j(y) dx \xrightarrow{\text{cancel}} \frac{dy}{dx} j(y) = \frac{g(x)}{j(y)} j(y) dx$$

$\underbrace{\qquad \qquad \qquad}_{\text{multiply both sides of eqn}}$

variables are "separated": $j(y) dy = g(x) dx$

integrate:

$$\int j(y) dy = \int g(x) dx$$

antiderivatives:

$$J(y) + C_1 = G(x) + C_2$$

$$J(y) = G(x) + \underbrace{C_2 - C_1}_{\equiv C}$$

only difference matters

implicit soln!

$$J(y) = G(x) + C$$

If possible, isolate y by inverting function J :

explicit soln

$$y = J^{-1}(G(x) + C)$$

CHECK: use implicit differentiation:

$$\frac{d}{dx} J(y) = \frac{d}{dx} (G(x) + C)$$

$$J'(y) \frac{dy}{dx} = G'(x)$$

$$j(y) \frac{dy}{dx} = g(x)$$

$$\frac{dy}{dx} = \frac{g(x)}{j(y)} \quad \checkmark \quad \text{yes, it works!}$$

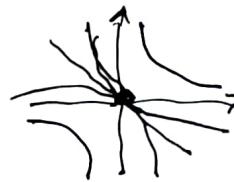
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Separable DEs

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example: $\left[\frac{dy}{dx} = -\frac{y^2}{x^2} \right]$

$\downarrow \star \left(\frac{dx}{y^2} \right)$ multiply both sides



direction field
last class

$$\frac{dy}{y^2} = -\frac{dx}{x^2}$$

\Leftarrow differentials must be in numerator!

$$\int \frac{dy}{y^2} = -\int \frac{dx}{x^2} \rightarrow \int y^{-2} dy = -\int x^{-2} dx \quad \text{power rule}$$

$$\frac{y^{-1}}{-1} = -\left(\frac{x^{-1}}{-1}\right) + C_1 \quad \leftarrow \text{only need one constant!}$$

$\star(-1)$

$$\frac{1}{y} = \frac{1}{x} + C_1 \quad \leftarrow \text{set } C_1 = -C$$

(implicit soln:

$$\boxed{\frac{1}{x} + \frac{1}{y} = -C_1 = C}$$

(simpler,
not necessary)

isolate y:

$$\frac{1}{y} = C - \frac{1}{x} = \frac{Cx-1}{x}$$

explicit soln:

$$\boxed{y = \frac{x}{Cx-1}}$$

initial condition: $y(1) = 2$ or $x=1, y=2$

backsub: $\frac{1}{1} + \frac{1}{2} = C$

$\hookrightarrow C = 3/2$ \leftarrow never last step? don't BOX!

no need to plug
into final expression
plug into
expression solving C!

backsub:

$$\boxed{y = \frac{x}{\frac{3}{2}x-1} = \frac{2x}{3x-2} = \frac{2x/3}{x-2/3}}$$

"simplest"

shows vert. asympt.
at $x = 2/3$

NOTE: "missing soln" $y=0$ not in this family!

division by y^2 prevents rest of derivation from allowing $y=0$!

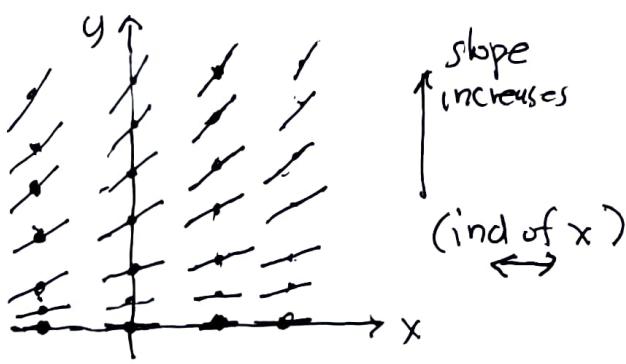
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Separable DES

example: power notation!

$$\frac{dy}{dx} = 2\sqrt{y} = 2y^{1/2}$$

$y=0$ obvious soln
(x-axis)



keep factors on RHS!

separate: $\frac{dy}{y^{1/2}} = 2 dx$

integrate: $\int y^{-1/2} dy = 2 \int dx$

$$\frac{y^{1/2}}{\frac{1}{2}} = 2x + C_1$$

$$y^{1/2} = \frac{1}{2}(2x + C_1) = x + \frac{C_1}{2} \quad \text{rename constant (simpler)}$$

$y^{1/2} = x + C$ implicit soln $x + C \geq 0 \rightarrow x \geq -C$ only

$y = (x + C)^2$ explicit soln, BUT $x \geq -C$ AND
↳ $x = -C$ is vertex of parabola

$y = 0$ for
 $x \leq -C$

initial condition off x-axis

$y(1) = 4$ or $x=1, y=4$: no need to go backwards
solving for C from end result

$$4^{1/2} = 1 + C$$

$$\hookrightarrow C = 2 - 1 = 1 \rightarrow \begin{cases} y = (x+1)^2 & \text{for } x \geq -1 \\ 0 & \text{for } x \leq -1 \end{cases}$$

NOTE: $f(x,y) = 2y^{1/2}$ NO x !
 $f(x,y) = x^2$ NO y ! } both still "separable"

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Separable DEs

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example (exponential behavior)

$$\frac{dy}{dx} = ky$$

$$\int \frac{dy}{y} = \int k dx$$

P
E
V
S
I
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$$|\ln y| = kx + C_1$$

implicit soln

Maple does not insert abs value signs because $\ln(-2)$ is complex—
Maple correct over complex numbers!

Isolate y:

$$e^{|\ln y|} = e^{kx+C_1} \quad \begin{matrix} \downarrow \\ "y" \end{matrix} \quad = e^{C_1} e^{kx} \quad (\text{rules of exponents})$$

$$y = \underbrace{\pm e^{C_1}}_C e^{kx} \quad (\text{if } |y|=Q, y=\pm Q!)$$

y cannot equal 0
until this point

redefine constant (simpler)
but also now includes $C=0$ soln
 $y=0$ while no real value of C_1
yields this "missing soln"

$$y = C e^{kx}$$

general soln

$$y_0 \equiv y(0) = C e^0 = C \quad (\text{constant is initial value at } x=0)$$

$$y = y_0 e^{kx}$$

soln of IVP with $y(0)=y_0$

exponential behavior very important

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example $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$ ← problems when $3y^2-5=0$
 $\hookrightarrow y^2 = 5/3, y = \pm(5/3)^{1/2}$
 vertical slope at these y values

$$\int (3y^2-5) dy = \int (4-2x) dx \quad (\text{separate & integrate})$$

$$y^3 - 5y = 4x - x^2 + C$$

$$y^3 - 5y - 4x + x^2 = C \quad (\text{implicit solution})$$

implicitplot this function to see solution curves.

see Maple

Final Remark: Some RHS functions must be factored FIRST!

$$\frac{dy}{dx} = y e^{x+y} = y e^x e^y = (e^y) e^x$$

$$\frac{dy}{dx} = \frac{xy+yt-x-1}{(x+1)y-(x+1)} = (x+1)(y-1)$$

or even solve first for $\frac{dy}{dx}$:

$$x^2 \frac{dy}{dx} = 1-x^2 + \underbrace{y^2 - x^2 y^2}_{(1-x^2)y^2} = (1-x^2)(1+y^2)$$

↓

$$\frac{dy}{dx} = \frac{1-x^2}{x^2} (1+y^2)$$