

## 1.2 Integration = Solving a DE

①

Integration has 2 aspects:

- 1) Finding a particular antiderivative.
- 2) Finding all antiderivatives.

$$f(x) \xrightarrow{\substack{\text{antiderivative} \\ \text{indefinite} \\ \text{integration}}} F(x) \text{ such that } F'(x) = f(x)$$

$$\int f(x) dx = F(x) + C$$

arbitrary constant,  
can take any real value

### General 1st order DE solvable for derivative

$$\frac{dy}{dx} = f(x, y) \quad \left\{ \begin{array}{l} \text{where } f \text{ is a function of points } (x, y) \text{ in the} \\ \text{x-y plane — interpretation: } f(x, y) \text{ is the slope} \\ \text{of a soln curve pt } (x, y) \text{ which passes thru } (x, y). \end{array} \right.$$

[counter-example:  $(y')^5 + xy y' = x^2$  cannot be solved  
for  $y'$  to put into this form]

special case:

$$\frac{dy}{dx} = f(x)$$

$\leftarrow$  RHS (right-hand side) does not depend on  $y$   
consequence: changing  $y$  (moving vertically in x-y plane)  
doesn't change DE, slope remains same →  
slope of soln curves unchanged under vertical  
translation

sln by  
indefinite  
integration

$$\begin{aligned} \underbrace{\int \frac{dy}{dx} dx}_{= \int \frac{dy}{dx} dx} &= \underbrace{\int f(x) dx}_{= F(x) + C_2} \\ &= \int 1 dy \quad \text{where } F'(x) \approx f(x) \\ &= y + C_1 \end{aligned}$$

$$\therefore \begin{aligned} y + C_1 &= F(x) + C_2 \\ y &= F(x) + \underbrace{C_2 - C_1}_{\equiv C} \end{aligned}$$

only difference  
of 2 constants  
matters,  
rename

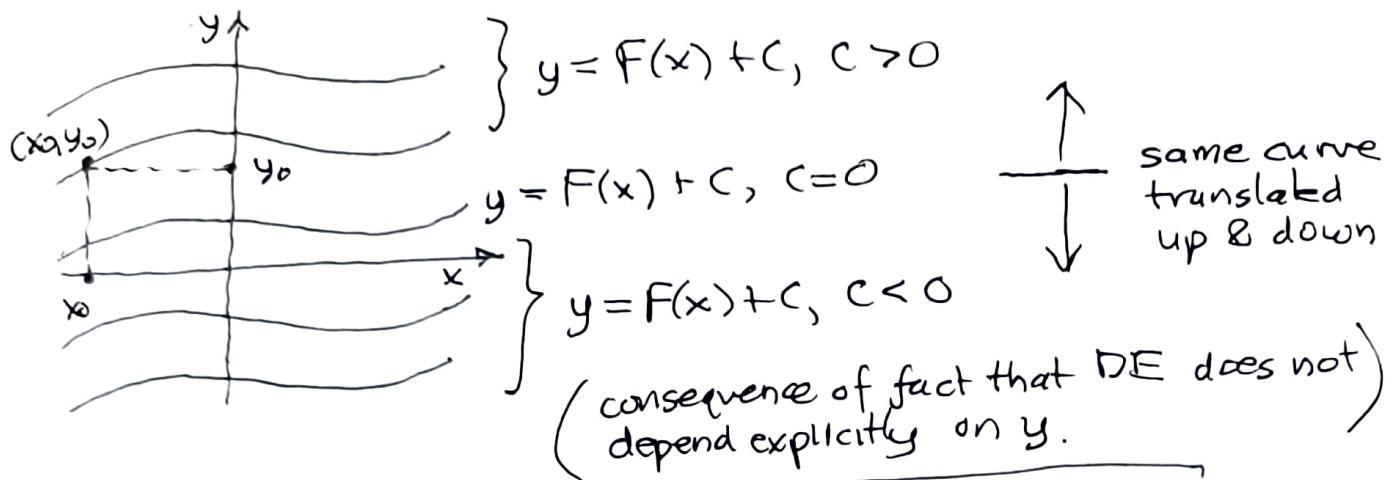
$$y = F(x) + C$$

general  
solution

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Interpretation (graphical):



IVP (initial value problem):

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

Find soln curve passing through point  $(x_0, y_0)$ .

once antiderivative found, fix C :

$$y_0 = y(x_0) = F(x_0) + C$$

$$\hookrightarrow C = y_0 - F(x_0)$$

$$y = \underbrace{F(x)}_{\int_{x_0}^x f(u) du} + F(x_0) + y_0$$

$$y = \int_{x_0}^x f(x) dx + y_0$$

change to dummy  
variable u to avoid  
confusion

definite integral form of soln which

can be useful when simple antiderivative cannot be found!

$$y = \int_{x_0}^x f(u) du + y_0$$

Recall Day 2:  $\frac{dy}{dx} = 3 \sin 2x, y(0) = -1$

$$y = \int 3 \sin 2x dx = -\frac{3}{2} \cos 2x + C$$

$$-1 = y(0) = -\frac{3}{2} \cos 2(0) + C = C - \frac{3}{2} \rightarrow C = \frac{3}{2} - 1 = \frac{1}{2} \rightarrow y = \frac{1}{2} - \frac{3}{2} \cos 2x$$

**OR**

$$y = \int_0^x 3 \sin 2u du - 1 = -\frac{3}{2} \sin 2u \Big|_0^x - 1 = -\frac{3}{2} \sin 2x + \frac{3}{2} - 1$$

$$= \frac{1}{2} - \frac{3}{2} \sin 2x \quad \checkmark$$

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successive integration (repeat!) iterates this procedure

- $\frac{d^2y}{dx^2} = f(x) \Leftrightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$

integrate:  $\frac{dy}{dx} = \underbrace{\int f(x) dx}_{F(x) + C_1} = F(x) + C_1$

again:  $y = \underbrace{\int (F(x) + C_1) dx}_{\text{antiderivative}} = G(x) + C_1 x + C_2$

now 2 arbitrary constants

- $\frac{d^n y}{dx^n} = f(x) \rightarrow$  leads to  $n$  arbitrary constants

↑ signals  $n$ th order DE → in fact the general soln of ANY  $n$ th order DE has  $n$  arbitrary constants;

specifying  $n$  initial conditions for the general solution yields a specific solution

example: 1 dimensional motion (high school physics!)

position:  $x(t)$   
velocity:  $x'(t) = \frac{dx}{dt}(t) = v(t)$   
acceleration:  $x''(t) = \frac{d^2x}{dt^2}(t) = v'(t) \equiv a(t)$

} physical terminology and usual variable names

Newton's law:  $F(t) = m a(t) \rightarrow a(t) = F(t)/m$

$$\frac{dv}{dt} = \frac{F(t)}{m} \quad \begin{matrix} \downarrow \\ \text{1st order DE} \\ \text{for } v \end{matrix}$$

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m} \quad \begin{matrix} \downarrow \\ \text{2nd order DE} \\ \text{for } x \end{matrix}$$

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Constant acceleration case  $F(t) = F_0 \rightarrow a(t) = F_0/m = a_0$

$$\frac{dv}{dt} = a_0 \rightarrow v = \int a_0 dt = a_0 t + C_1$$
$$v(0) = 0 + C_1 \equiv v_0 \rightarrow C_1 = v_0 \rightarrow v = a_0 t + v_0$$

$$\frac{dx}{dt} = v = a_0 t + v_0 \rightarrow x = \int a_0 t + v_0 dt = \frac{1}{2} a_0 t^2 + v_0 t + C_2$$
$$x(0) = 0 + 0 + C_2 \equiv x_0 \rightarrow x = \frac{1}{2} a_0 t^2 + v_0 t + x_0$$

gen soln expressed in terms of initial conditions  
at  $t=0$ .

WHAT IS LEFT TO DO?

Finding a soln is just the first step to using that  
solution to ask various questions in physical applications.

The following lunar landing problem  
shows an interesting twist on this kind  
of application.

## Lunar Lander Problem E&P 2 , Example 1.2.2



$a \uparrow \downarrow v$

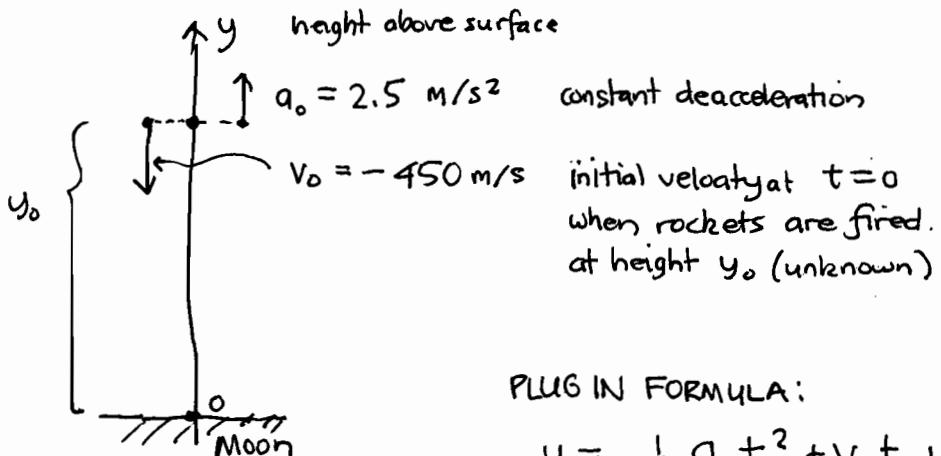
A lunar lander is freely falling toward the surface of the moon at a speed of 450 m/s.

Its retro rockets, when fired, provide a constant deceleration of 2.5 m/s<sup>2</sup> (the gravitational acceleration produced by the moon is assumed to be included in the given deceleration).

At what height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown" ( $v=0$  at impact)?

### Solution

- Draw a diagram, set up variables:



soft landing condition:

when  $v(t_1) = 0$ ,  
 $y(t_1) = 0$ .

PLUG IN FORMULA:

$$y = \frac{1}{2} a_0 t^2 + v_0 t + y_0$$

$$\downarrow \quad 2.5 \quad -450$$

$$y = \frac{1}{2} (2.5)t^2 - 450t + y_0$$

[WHEN]

$$v = \frac{dy}{dt} = (2.5)t - 450 = 0 \rightarrow t = \frac{450}{2.5} \equiv t_1$$

$$= 180 \text{ (sec)}$$

$$= 3 \text{ min}$$

$$0 = y(t_1) = \frac{1}{2}(2.5)(180)^2 - 450(180) + y_0$$

[WHERE]

$$y_0 = 450(180) - \frac{1}{2}(2.5)(180)^2$$

$$= 40500 \text{ m} = 40.5 \text{ km} (\approx 25.3 \text{ mi})$$

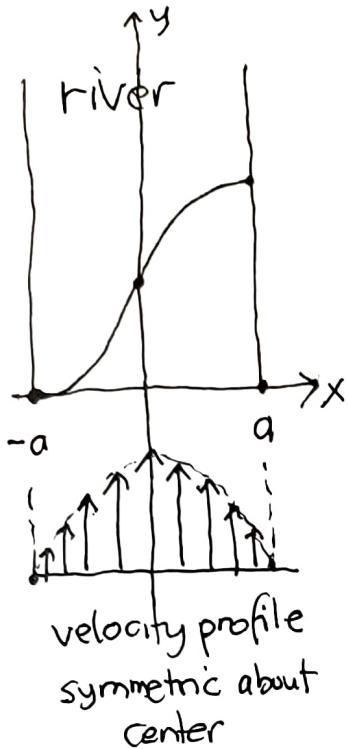
for metric  
challenged  
Americans

Final Answer: Fire the retrorockets at height 40.5 km above surface of moon.

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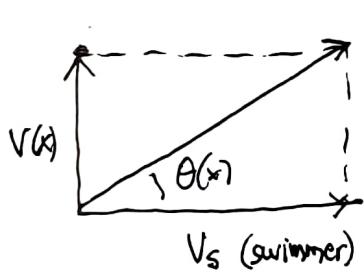
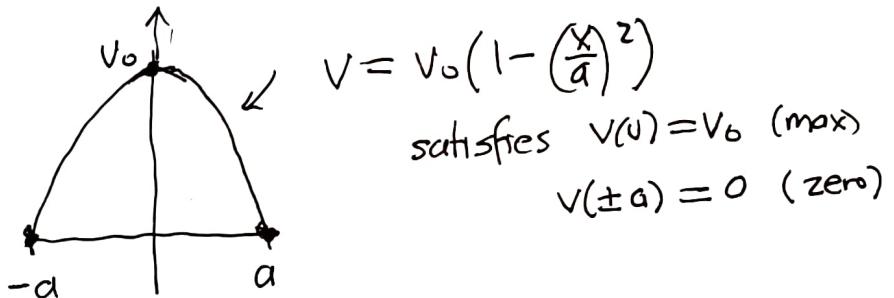
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Setting up a DE from a word problem / Example 1.2.4: A Swimmer's Problem



- straight river, parallel shores.
- current downstream strongest at center, symmetric about center, zero at shore
- swimmer aims for opposite shore but river drags downstream
- ▲ how far downstream does swimmer reach opposite shore

need model for velocity profile — easiest is quadratic (parabola thru 3 key points):



$$\text{slope} = \tan \theta(x) = \frac{V(x)}{V_s} = \frac{dy}{dx} \quad \begin{matrix} \text{tangent} \\ \text{to soln} \\ \text{curve} \end{matrix}$$

[See textbook setup]

Leads to a DE involving 3 parameters  $a, V_0, V_s$  with initial condition putting swimmer on left shore.

IVP

$$\left\{ \begin{array}{l} \frac{dy}{dx} = \frac{V_0}{V_s} \left(1 - \frac{x^2}{a^2}\right) \quad \text{DE} \\ y(-a) = 0 \quad \text{init} \end{array} \right.$$

Plug in numbers, integrate, impose initial condition, find  $y(x)$ , evaluate  $y(a)$  to get net displacement along shore. [see Maple]

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### BONUS

DEs with unspecified value parameters require additional conditions to give them concrete values.

#### example

The rate of change of Celsius temperature with respect to Fahrenheit temperature is constant.

$$\text{DE: } \frac{dC}{dF} = k \xrightarrow[\substack{\text{parameter in DE}}]{\substack{\text{gen} \\ \text{soln}}} C = kF + C_1 \xrightarrow[\substack{\text{soln parameter}}]{\uparrow}$$

$$\text{init: } C(32) = 0 \rightarrow 0 = 32k + C_1 \rightarrow C_1 = -32k$$

$$\text{extra: } C(212) = 100 \rightarrow 100 = 212k + C_1 \rightarrow 100 = (212-32)k = 180k$$

$$\rightarrow k = \frac{100}{180} = \frac{5}{9}$$

$$\rightarrow C_1 = -32\left(\frac{5}{9}\right)$$

$$C = \frac{5}{9}F - 32\left(\frac{5}{9}\right) = \frac{5}{9}(F-32)$$

invert  $\downarrow$  "subtract 32, multiply by  $5/9$ "

$$F = 32 + \frac{9}{5}C$$

I still have trouble making these conversions in my head!