

1.1.6 Differential Equations: Initial Conditions

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Initial conditions nail down solutions to DEs with arbitrary constants.

You already know how to solve a class of DEs.

When you antidifferentiate, you are solving a first order DE.

"Indefinite integration" provides the general solution.

$$\frac{dy}{dx} = f(x) \longrightarrow \text{soln: } y = \int f(x) dx = \underbrace{F(x)}_{\substack{\text{particular} \\ \text{antiderivative} \\ F'(x) = f(x)}} + C \quad \begin{matrix} \uparrow \\ \text{arbitrary} \\ \text{additive} \\ \text{constant} \end{matrix}$$

The result is a family of solutions, one for each real value of C .

Example: $\frac{dy}{dx} = 3 \sin 2x \longrightarrow y = \int 3 \sin 2x dx = -\frac{3}{2} \cos 2x + C$

(if we want the soln for which $y(0) = -1$, then:

$$-1 = y(0) = C - \frac{3}{2} \cos 2(0) = C - \frac{3}{2} \longrightarrow C = \frac{3}{2} - 1 = \frac{1}{2} \longrightarrow y = \frac{1}{2} - \frac{3}{2} \cos 2x$$

HIGHER DERIVATIVES We can extend this to higher derivatives.

$$\begin{aligned} \frac{d^2y}{dx^2} = f(x) &\longrightarrow \frac{dy}{dx} = \int f(x) dx = F(x) + C_1 \\ &\longrightarrow y = \int F(x) + C_1 dx = \underbrace{G(x)}_{G' = F} + C_1 x + C_2 \end{aligned}$$

Each time we integrate indefinitely, we introduce another additive constant.

Example: $\frac{d^2y}{dt^2} = -32$ $y = \text{height in ft in constant grav. field}$

$$\hookrightarrow \frac{dy}{dt} = -32t + C_1 \longrightarrow y = -16t^2 + C_1 t + C_2$$

We need 2 conditions to get a particular trajectory, (at rest at $y=20$)
say $y(1) = 20, y'(1) = 0$, so:

$$\begin{aligned} 20 &= -16(1)^2 + C_1(1) + C_2 = -16 + 32 + C_2 \longrightarrow C_2 = 20 + 16 - 32 = 4 \\ 0 &= C_1 - 32(1) \longrightarrow C_1 = 32 \quad \therefore y = -16t^2 + 32t + 4. \end{aligned}$$

Similarly $\frac{d^ny}{dx^n} = f(x) \longrightarrow n$ arbitrary constants in solution.

This is true in general. Any n th order DE has n arbitrary constants in its general solution.

1.1b) Differential Equations: Initial Conditions

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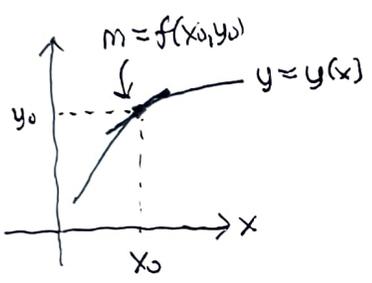
CHAPTER 1: First Order DEs.

We only consider DEs where we can isolate the highest derivative in the DE.

$(\frac{dy}{dx})^5 + \frac{dy}{dx} = xy \rightarrow$ we cannot solve this equation for $\frac{dy}{dx}$. FORGET IT.

We only consider DEs of the form with an initial condition $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ "initial value problem" (IVP)

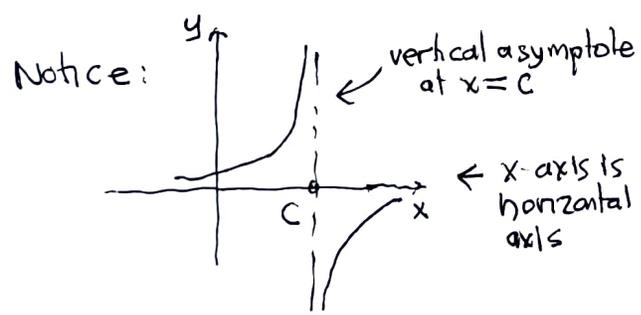
gives slope of graph of a solution at point (x,y) in plane



A solution of an IVP like this is a particular curve which passes through the point (x_0, y_0) in the plane.

Example. $\frac{dy}{dx} = y^2$, $\begin{cases} a) y(1) = 2 \\ b) y(1) = 0 \end{cases}$

Check soln: $y = \frac{1}{c-x} = (c-x)^{-1}$
 $\frac{dy}{dx} = -(c-x)^{-2}(-1) = \frac{1}{(c-x)^2}$
 $\frac{dy}{dx} = y^2 \rightarrow \frac{1}{(c-x)^2} = (\frac{1}{c-x})^2 = (\frac{1}{c-x})^2$



a) $2 = y(1) = \frac{1}{c-1} \rightarrow c-1 = \frac{1}{2} \rightarrow c = 1 + \frac{1}{2} = \frac{3}{2} \rightarrow y = \frac{1}{\frac{3}{2} - x} (= \frac{2}{3-2x})$

b) $0 = y(1) = \frac{1}{c-1}$ SDP, no solution BUT $y=0$ (constant function) is a solution since $\frac{dy}{dx} = 0$ so $0 = 0^2 = 0 \checkmark$

The general solution formula omits all solutions for initial data on the x-axis. Solution techniques can yield formulas which omit solutions. We need to understand when solutions to an IVP exist and if they are unique etc. Otherwise we can miss solutions.

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Simple intuitive application: Population dynamics

P = population number (people, bacteria, whatever)

growth without constraints should be roughly proportional to population number P
(twice the people, twice the babies!)

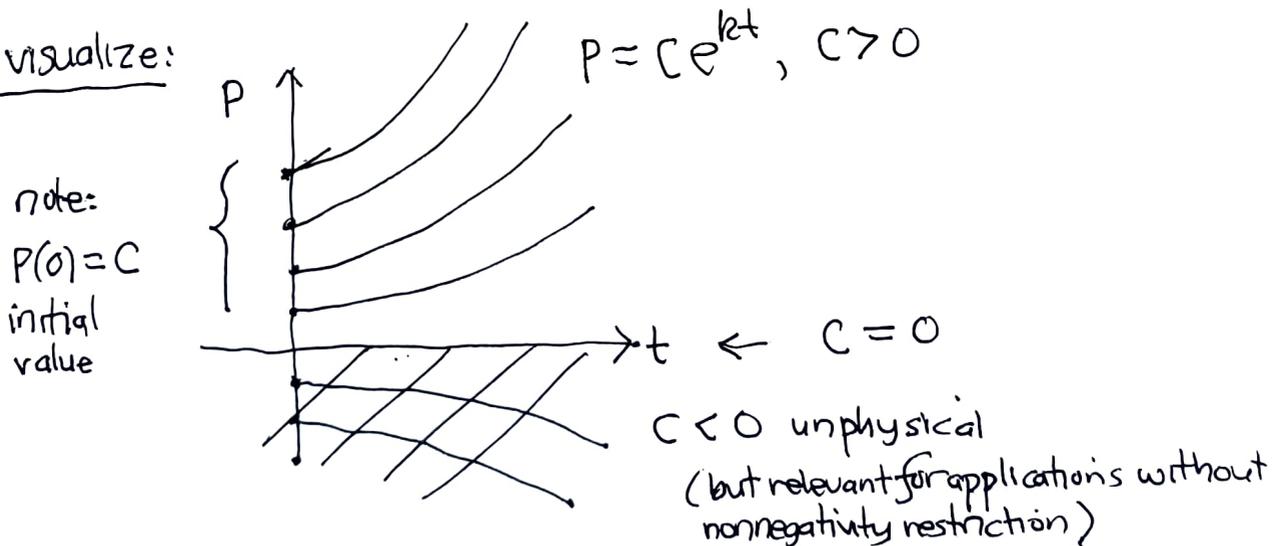
$\frac{dP}{dt} \propto P \rightarrow \boxed{\frac{dP}{dt} = kP} \geq 0$ (growth) $\rightarrow k > 0$ "DE with parameter"
need extra condition to nail down rate of growth k ($\frac{1}{k}$ has units of time!)
Note: $P \geq 0$!

time in appropriate units \rightarrow

general soln: $P(t) = C e^{kt}$
 C arbitrary constant for family of solns
exponential rate factor already in DE.

check: $\frac{dP}{dt} = \frac{d}{dt}(C e^{kt}) = C \frac{d}{dt} e^{kt} = C e^{kt} \frac{d}{dt}(kt) = k C e^{kt}$
 $\frac{dP}{dt} = kP \rightarrow k C e^{kt} = k(C e^{kt}) \checkmark$

visualize:

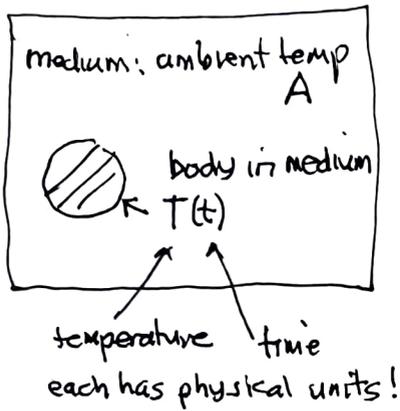


BUT for a real population limits on growth require more complicated model
model can be valid for a limited time under the right circumstances

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Application: Newton's law of cooling/heating: simple modeling



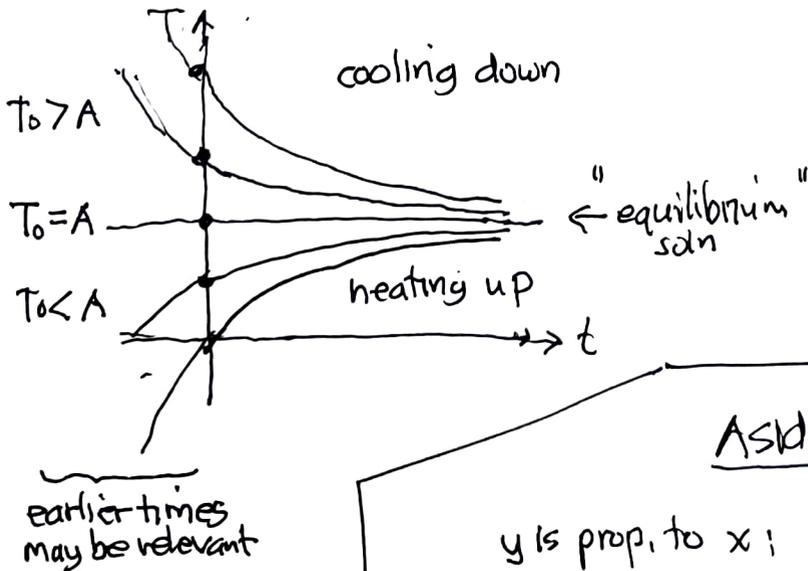
- food cools much quicker with liquid nitrogen than just insertion into freezer
- ice cube melts much faster in an oven than on the kitchen table
- ∴ time rate of change of temperature should be proportional to the temperature difference (simplest assumption to make math model!)

$$\frac{dT}{dt} \propto (T-A)$$

simplest math model \rightarrow IVP

$$\left\{ \begin{array}{l} \frac{dT}{dt} = -k(T-A), \quad k > 0 \\ < 0 \text{ if } T > A \text{ (cooling down!)} \\ > 0 \text{ if } T < A \text{ (heating up!)} \\ T(0) = T_0 \text{ initial condition at } t=0 \\ \text{(} T(t_0) = T_0 \text{ with } t_0 \neq 0 \text{ might be useful!)} \end{array} \right.$$

what we expect:



we will see this is an exponential decay process.
model fits reality well under certain conditions.

Aside: proportionality

y is prop. to x: $y \propto x \rightarrow y = kx$

y is inversely prop to x: $y \propto \frac{1}{x} \rightarrow y = \frac{k}{x}$

y is inversely prop to x^2 : $y \propto \frac{1}{x^2} \rightarrow y = \frac{k}{x^2}$