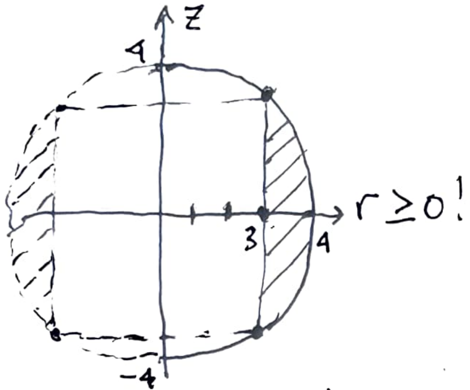
Wedding ring setup (1)

Drill a 3 unit radius hole through the center of a 4 unit radius sphere.
Set up with center at origin.

sphere: $x^2+y^2+z^2=4^2 \longrightarrow \rho^2=4^2 \rightarrow \rho=4$

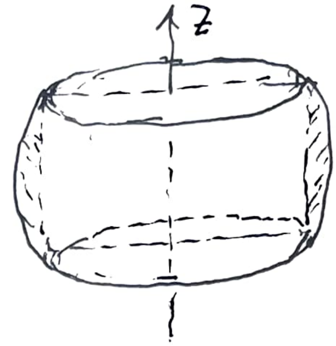
cylinder (hole): $x^2+y^2=3^2 \longrightarrow r^2=3^2 \rightarrow r=3$

$\rightarrow r^2+z^2=4^2$



revolved around to rest of plane (mirror reflection)

r-z halfplane (at fixed angle θ)



solid of revolution

$0 \le \theta \le 2\pi$

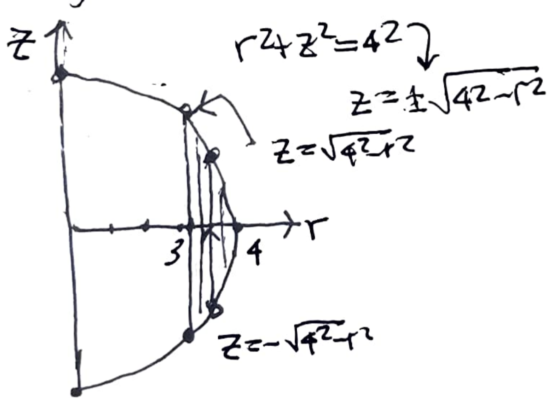
integrate last in triple integral

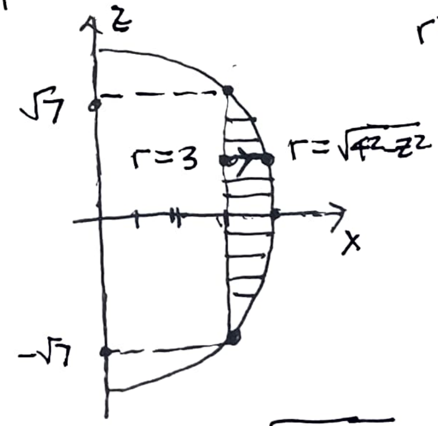
intersection points:

$r^2+z^2=4^2 \rightarrow z^2=4^2-3^2=7 \rightarrow z=\pm\sqrt{7}$

$\uparrow r=3$

Integration order in r-z half plane: z-first (vertical) or r-first (horizontal)

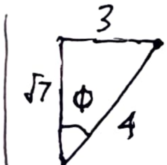




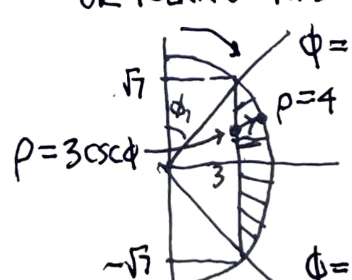
$r^2+z^2=4^2$
$r=\sqrt{4^2-z^2} \ge 0$

$z=-\sqrt{4^2-r^2} \,..\, \sqrt{4^2-r^2}$
while $r=3 \,..\, 4$
vertical linear cross-section ⇒ cylinder

$r=3 \,..\, \sqrt{4^2-z^2}$
while $z=-\sqrt{7} \,..\, \sqrt{7}$
horizontal linear cross-section ⇒ annulus



OR POLAR COORDS in r-z halfplane (spherical coords)



$\phi=\phi_1$ (acute)

$\phi=\phi_2=\pi-\phi_1$ (supplementary angles)

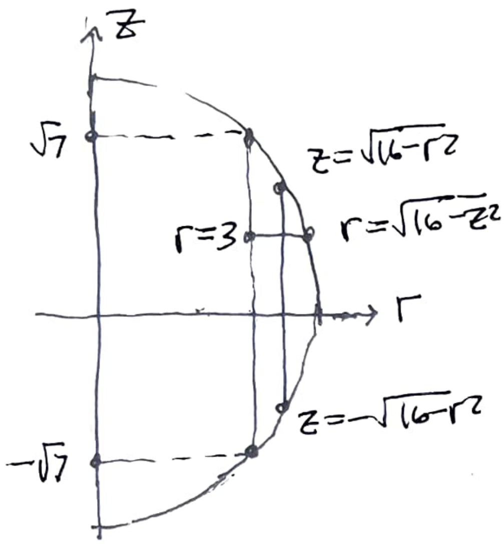
$\rho=3\csc\phi \,..\, 4$
while $\phi=\arcsin\frac{3}{4} \,..\, \pi-\arcsin\frac{3}{4}$
while $\theta=0 \,..\, 2\pi$

$z=\rho\cos\phi$
$r=\rho\sin\phi=3$
$\frac{r}{z}=\tan\phi$

$\rho=3\csc\phi$

$\phi=\arctan\left(\frac{3}{\sqrt{7}}\right)=\arccos\left(\frac{\sqrt{7}}{4}\right)$
$=\arcsin(3/4)$

Wedding Ring setup (2): integration



inner double integral diagram

z-first (vertical)

$$\int_0^{2\pi}\int_3^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} f\ \underbrace{r\,dz\,dr\,d\theta}$$

always do θ integral last ↓

$dV = dz\,dr\,(r\,d\theta)$

nothing depends on θ, factors out, do at any time $\left(\int_0^{2\pi} 1\,d\theta = 2\pi\right)$ ← (if revolved 360°, otherwise limit range of θ)

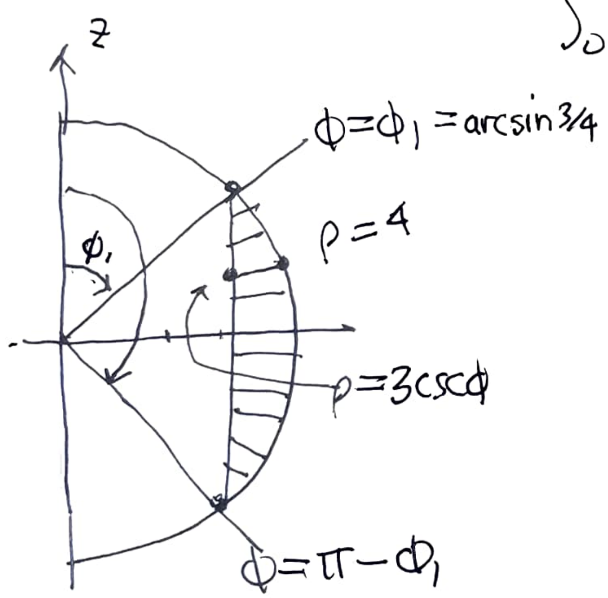
r-first (horizontal)

$$\int_0^{2\pi}\int_{-\sqrt{7}}^{\sqrt{7}}\int_3^{\sqrt{16-z^2}} f\ \underbrace{r\,dr\,dz\,d\theta}$$

$dV = (dr)(dz)(r\,d\theta)$

$\underbrace{d\rho(\rho\,d\phi)\ (\rho\sin\phi)\,d\theta}$

$\rho^2\sin\phi\ d\rho\,d\phi\,d\theta$



$\rho^2\sin\phi$ ↙ geometric correction factor $= \rho\cdot r$ product of two arc radii converting $d\phi$ and dr to arclength differentials.

$d\rho$ ↑ do ρ first (like do r first)

ρ-first (radial)

$$\int_0^{2\pi}\int_{\arcsin 3/4}^{\pi-\arcsin 3/4}\int_{3\csc\phi}^{4} f\ \rho^2\sin\phi\ d\rho\,d\phi\,d\theta$$

$d\rho$: radial integral always done first

since $\rho=\rho(\phi)$ always defines curves of variable ϕ.

Cartesian to cylindrical / spherical integration

convert $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz\, dy\, dx \equiv Q$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{2-x^2-y^2}} xy \, dz\, dy\, dx$$

$z^2 = 2-x^2-y^2$

$\underbrace{x^2+y^2}_{r^2} + z^2 = 2 = \rho^2$

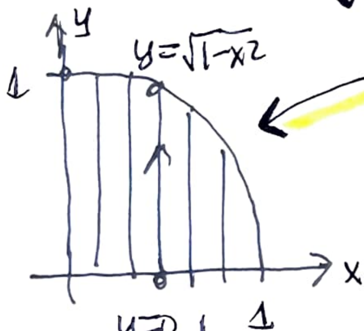
$r^2+z^2=2 \to z = \pm\sqrt{2-r^2}$

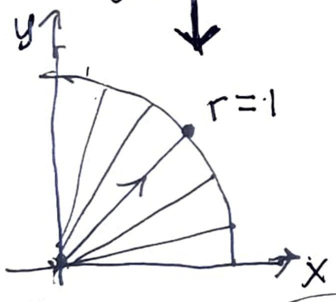
$y^2 = 1-x^2$

$x^2+y^2=1$

$z^2 = x^2+y^2$ cone

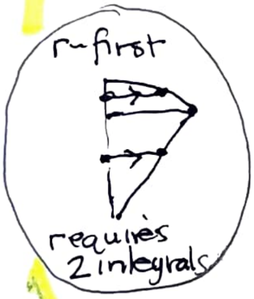
$z = r$

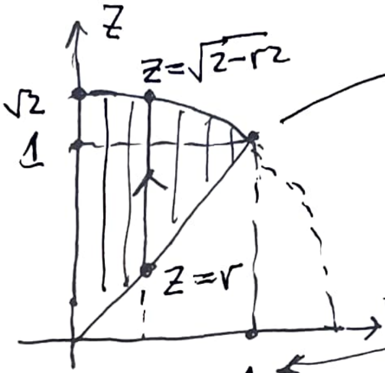




$r = 0..1$ while $\theta = 0..\frac{\pi}{2}$

outer integral





intersect:

$r = \sqrt{2-r^2}$

$r^2 = 2-r^2 \to r^2=1$

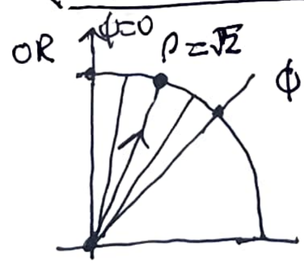
$r=1$

$z=r=1$

inner double integral

$z = r..\sqrt{2-r^2}$ while $r = 0..1$

OR



$\phi = \pi/4 \leftarrow (\tan\phi = 1)$

$\rho = 0..\sqrt{2}$ while $\phi = 0..\pi/4$.

3d!

1/4 of a snow cone!

$dA = r\,dr\,d\theta$

$$Q = \int_0^{\frac{\pi}{2}} \int_0^1 \int_r^{\sqrt{2-r^2}} \underbrace{r\cos\theta}_{x}\,\underbrace{(r\sin\theta)}_{y}\; r \; dz\, dr\, d\theta$$

3 factors of r

$d\rho\,(\rho\, d\phi)$

$dV = r\rho\, d\rho\, d\phi\, d\theta$

$\rho^2\sin\phi$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} (\rho\sin\phi)^3 \; \theta\sin\theta\, \rho \; d\rho\, d\phi\, d\theta$$

$\underbrace{(\rho\sin\phi\cos\theta)}_{x}\,\underbrace{(\rho\sin\phi\sin\theta)}_{y}\,(\rho^2\sin\phi)$