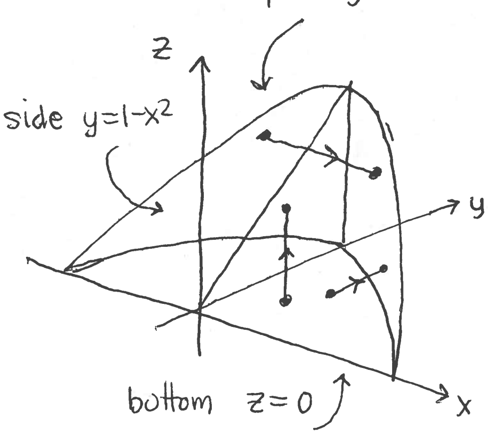
deciphering triple integral iteration: divide & conquer



innermost limits of integration — we need starting and stopping values of the innermost ("first") variable of integration as functions of the two remaining outer variables of integration

z-first: we integrate in the z-direction from the bottom ($z=0$) to the top ($z=y$) so

$z = 0 .. y \rightarrow \int_0^y f(xyz)\,dz$

y-first: we integrate in the y-direction from the top ($z=y$) to the side ($y=1-x^2$) so

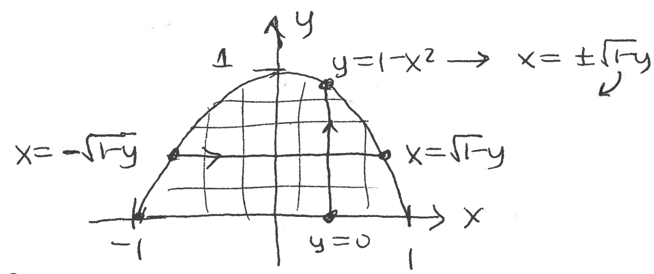
$y = z .. 1-x^2 \rightarrow \int_z^{1-x^2} f(xyz)\,dy$

x-first: we integrate in the x-direction from the back wall ($y=1-x^2, x\le 0$) to the front wall ($y=1-x^2, x\ge 0$) looking down the x-axis. Solving for the two x-values we get

$x = -\sqrt{1-y} .. \sqrt{1-y} \rightarrow \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(xyz)\,dx$

outer double integral iteration

once we've done the innermost integral, we collapse the figure to a region of the plane of the other two variables bounded by the intersections of the surfaces found by eliminating the innermost variable from the pair of equations describing the two intersecting surfaces for each part of the edge.



z-first case

top $z=y$ & bottom $z=0$ intersect at $y=0$

top $z=y$ & side $y=1-x^2$ intersect at $y=1-x^2$

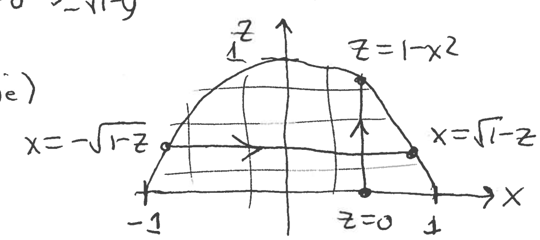
y-first then x: $y=0..1-x^2$ while $x=-1..1 \longrightarrow \int_{-1}^1\int_0^{1-x^2}(\ldots)\,dy\,dx$

x-first then y: $x=-\sqrt{1-y} .. \sqrt{1-y}$ while $y=0..1$ $\quad \int_0^1\int_{-\sqrt{1-y}}^{\sqrt{1-y}}(\ldots)\,dx\,dy$

y-first case

top $z=y$ & bottom $z=0$ still intersect at $z=0$ (eliminate y this time)

top $z=y$ & side $y=1-x^2$ intersect at $z=1-x^2$ (eliminate y)

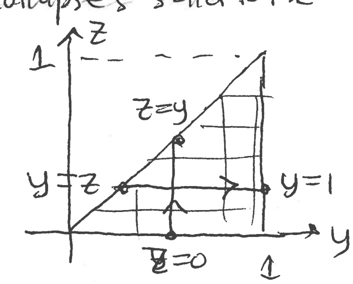


same as before with $y\to z$:

$\int_{-1}^1\int_0^{1-x^2}(\ldots)\,dz\,dx = \int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}(\ldots)\,dx\,dz$ ← y-first single integral from above

x-first case

$y=1-x^2 \rightarrow x=\pm\sqrt{1-y}$ describes 2 halves of the sidewall. Ignoring x collapses solid to the triangular region of the y-z plane.



z-first then y: $z=0..y$ while $x=0..1 \rightarrow \int_0^1\int_0^y(\ldots)\,dz\,dy$

y-first then z: $y=z..1$ while $z=0..1 \rightarrow \int_0^1\int_z^1(\ldots)\,dy\,dz$

x-first single integral from above