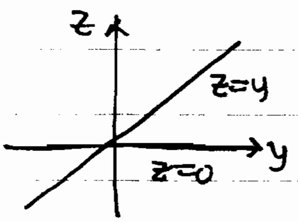
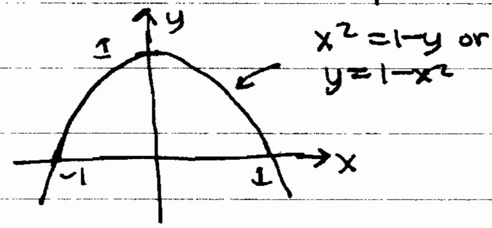
Exercise in setting up triple integrals in Cartesian coordinates

Set up a triple integral over the region enclosed by the surfaces $z=0$, $z=y$, $x^2=1-y$ for any function $f(x,y,z)$ in all 6 possible iterations (15.7.29)

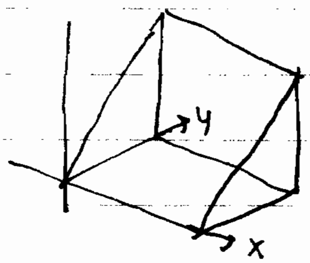
How to visualize the region

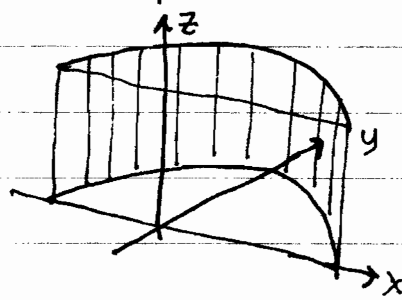
The first 2 surfaces are curves in the y-z plane, the last a curve in the xy plane:

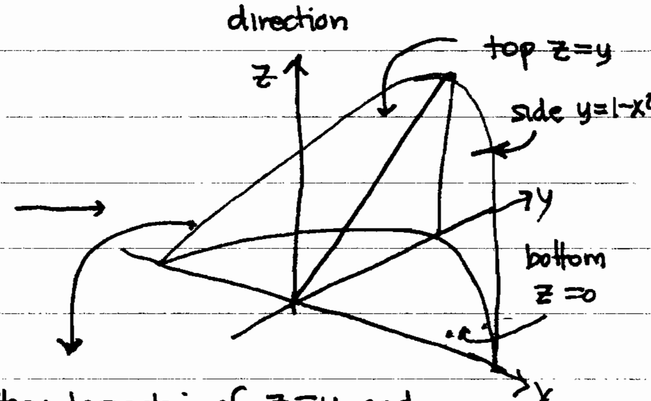




These become surfaces by extending them in the third direction

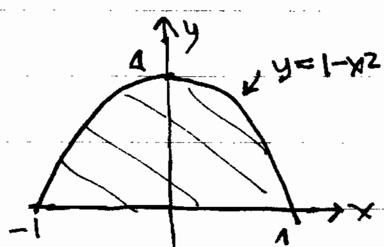


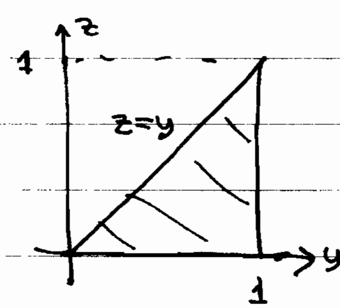


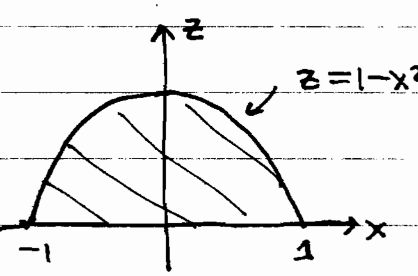


the intersection of $z=y$ and $y=1-x^2$ can also be described by $z=y$ and $z=1-x^2$, giving the projection of the solid onto the xz plane.

We now see the projections of this solid onto the 3 coordinate planes, each of which corresponds to the region of the outer double integral which can be iterated in two ways, while the inner integral is in the third coordinate which has to be given a starting and stopping value







z from:
to:

x from:
to:

y from:
to:

In each case put in the 2 cross-sections (labeled endpoints) for the double integrals, identify the starting and stopping equations for z, then write down the 2 triple integrals.
You can check by setting $f(xyz)=1$ and integrating (MAPLE is quicker).

6 way iteration example

tetrahedron:

$x=1,\ y=2,\ z=3$

$\frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1$

solve for each variable in turn:

$x=2\left(1-\frac{y}{4}-\frac{z}{6}\right)$

$y=4\left(1-\frac{x}{2}-\frac{z}{6}\right)$

$z=6\left(1-\frac{x}{2}-\frac{y}{4}\right)$

starting values for 3 partial integrations

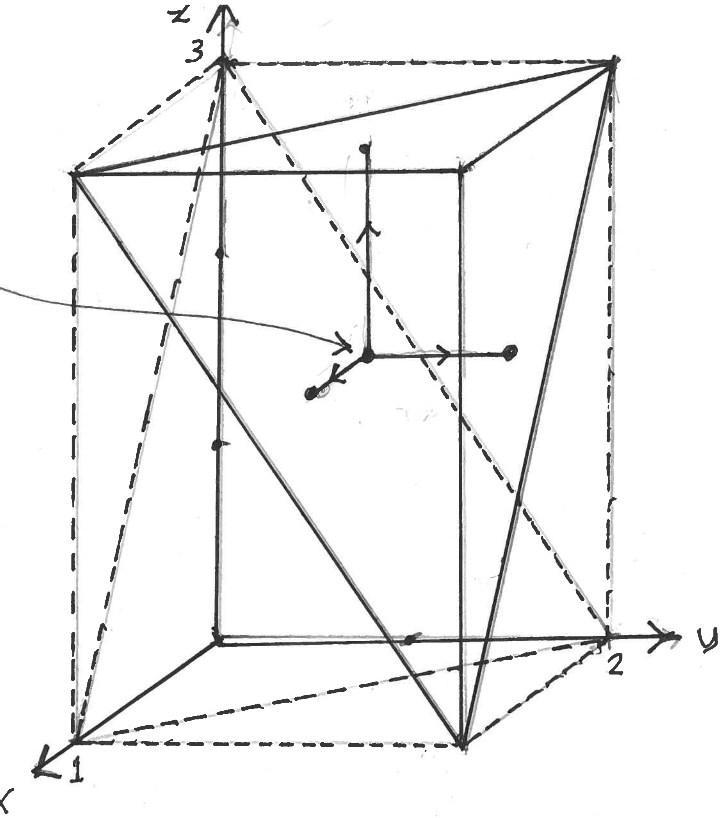
ending values:

$x=1$

$y=2$

$z=3$

these give lower & upper limits for innermost integration



innermost integral moves along coord axes from the oblique plane outward from origin

outer double integral is done over projection of solid to coordinate planes of remaining variables

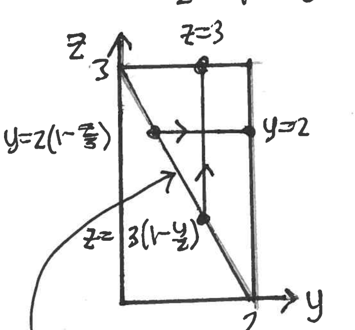
dashed triangles are those projections.

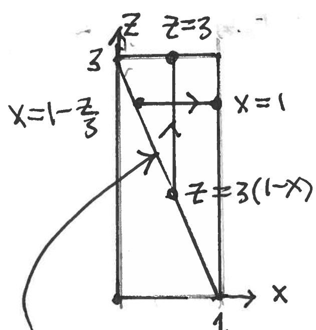
intersections of faces are common solutions of pairs of equations, eliminate one variable:

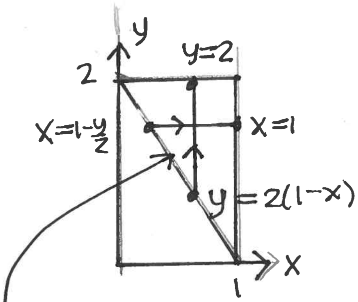
$x=1 \to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{1}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{y}{2}+\frac{z}{3}=1$ project solid onto yz plane

$y=2 \to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{x}{2}+\frac{2}{4}+\frac{z}{6}=1 \to x+\frac{z}{3}=1$ project solid onto xz plane

$z=3 \to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{x}{2}+\frac{y}{4}+\frac{3}{6}=1 \to x+\frac{y}{2}=1$ project solid onto xy plane







$\frac{y}{2}+\frac{z}{3}=1 \to z=3\left(1-\frac{y}{2}\right)$, $\to y=2\left(1-\frac{z}{3}\right)$

$x+\frac{z}{3}=1 \to z=3(1-x)$, $\to x=1-\frac{z}{3}$

$x+\frac{y}{2}=1 \to y=2(1-x)$, $\to x=1-\frac{y}{2}$

$$\left(\int_{2\left(1-\frac{y}{4}-\frac{z}{6}\right)}^{1} f\,dx\right) \qquad \left(\int_{4\left(1-\frac{x}{2}-\frac{z}{6}\right)}^{2} f\,dy\right) \qquad \left(\int_{6\left(1-\frac{x}{2}-\frac{y}{4}\right)}^{3} f\,dz\right)$$

$$\int_0^2\int_{3\left(1-\frac{y}{2}\right)}^{3} (\ \ )\,dz\,dy \qquad \int_0^1\int_{3(1-x)}^{3} (\ \ )\,dz\,dx \qquad \int_0^1\int_{2(1-x)}^{2} (\ \ )\,dy\,dx$$

$$\int_0^3\int_{2\left(1-\frac{z}{3}\right)}^{2} (\ \ )\,dy\,dz \qquad \int_0^3\int_{1-\frac{z}{3}}^{1} (\ \ )\,dx\,dz \qquad \int_0^2\int_{1-\frac{y}{2}}^{1} (\ \ )\,dx\,dy$$

when $f(xyz)=1$, all integrals give the volume: 1.