Center of mass, centroid (geometric center) (1)

mass moments

1-d

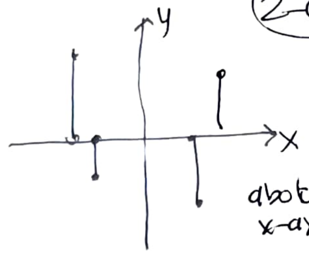
moment about origin, distance from $x=0$

product of masses times separation distances from point, axis, plane summed together.

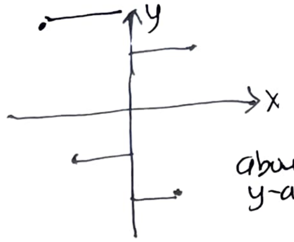
↓

continuous distributions integrate mass density against separations

2-d



about x-axis

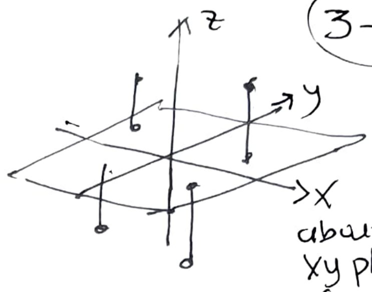


about y-axis

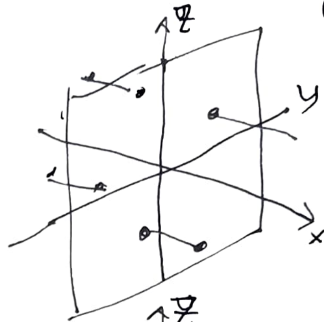
moments about axes

distances from $y=0$, $x=0$

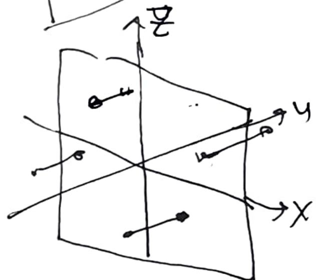
3-d



about xy plane ($z=0$)



about yz plane ($x=0$)



about xz plane ($y=0$)

distances from $z=0, y=0, x=0$

$M=\int \rho\, dx$ linear density: mass/length

$M=\iint \rho\, dA$ surface density: mass/area

$M=\iiint \rho\, dV$ volume density: mass/volume

3-d mass moments

$M_{yz}=\underbrace{\iiint \rho\, x\, dV}$ (about $x=0$), $M_{xz}=\iiint \rho\, y\, dV$ (about $y=0$), $M_{xy}=\iiint \rho\, z\, dV$ (about $z=0$)

center of mass:

weighted average of position vector against fractional mass density

$$\iiint\left(\frac{\rho}{M}\right)dV=\frac{\iiint \rho\, dV}{M}=\frac{M}{M}=1 \qquad \text{(similar in 1-d, 2-d)}$$

$$\langle\bar{x},\bar{y},\bar{z}\rangle=\iiint \langle x,y,z\rangle \frac{\rho}{M}\, dV=\frac{\iiint \langle x,y,z\rangle \rho\, dV}{\iiint \rho\, dV}$$

$\rho=\rho_0$ constant → ρ_0 cancels out of quotient, get geometric center = centroid

equivalent to set $\rho_0=1$:

$$\langle V, M_{yz}, M_{xz}, M_{xy}\rangle=\iiint \langle 1,x,y,z\rangle\, dV$$

divide these by first integral to get centroid

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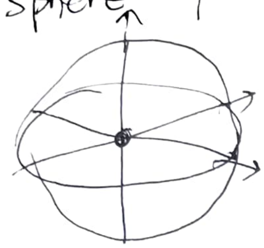
c.o.m.: $\langle\bar{x},\bar{y},\bar{z}\rangle=\dfrac{\langle M_{yz},M_{xz},M_{xy}\rangle}{M}$ $M \longrightarrow V$ for centroid

Center of mass, Centroid (2)

homogeneous sphere $\rho=\rho_0 \to$ centroid $=$ "center" of sphere (obvious!)

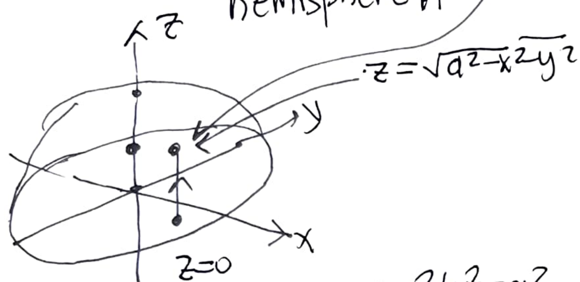


$\iiint x\,dV = 0$ since $x>0$ balances $x<0$ by symmetry

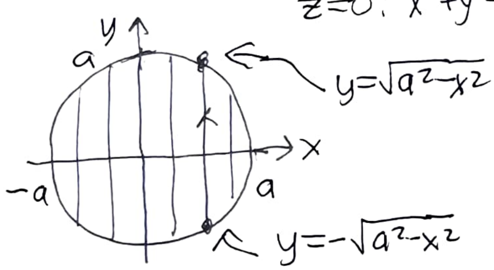
etc.

$x^2+y^2+z^2=a^2$, radius a.

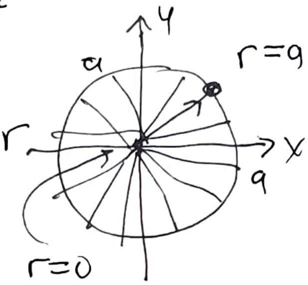
hemisphere H $\longrightarrow$ expect centroid to still be on z-axis but below halfway point to pole.



$z=0$: $x^2+y^2=a^2$



or



$$\iiint f\,dV$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} f\,dz\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} f\,r\,dz\,dr\,d\theta$$

result: $\langle \bar{x},\bar{y},\bar{z}\rangle = \langle 0,0,\frac{3a}{8}\rangle$

$= \langle 0,0,0.375a\rangle$

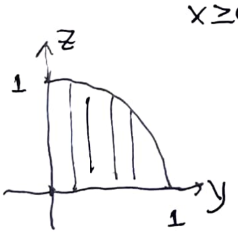
less than half as expected

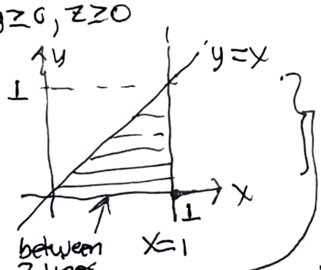
$\boxed{15.6.23}$

wedge cut from cylinder : $\rho=1 \to$ centroid

Region in first octant bounded by cylinder $y^2+z^2=1$ and the planes $y=x$, $x=1$.

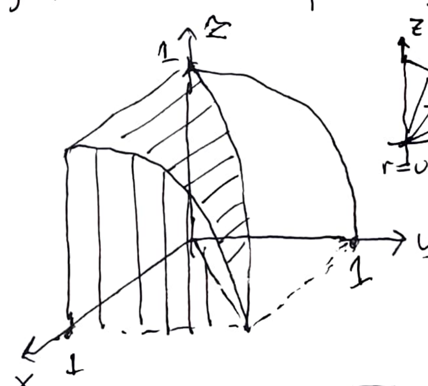
$x\ge 0, y\ge 0, z\ge 0$

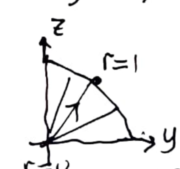




$0\le y\le 1$

$\rightarrow$

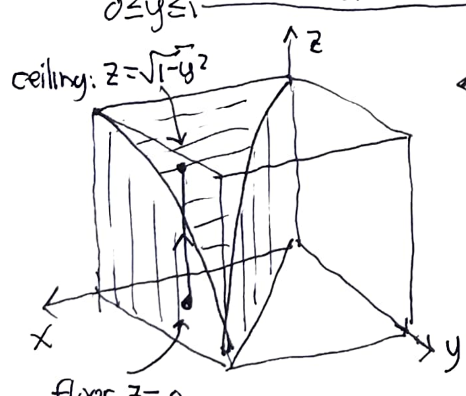


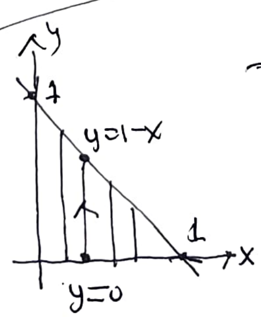


$y=r\cos\theta$

$z=r\sin\theta$

rotate view





$z=0..\sqrt{1-y^2}$ while $y=0..1-x$ while $x=0..1$

$$\iiint_R f\,dV = \int_0^1\int_0^{1-x}\int_0^{\sqrt{1-y^2}} f\,dz\,dy\,dx$$

OR $$=\int_0^1\int_0^{\sqrt{1-y^2}}\int_y^1 f\,dx\,dz\,dy$$

(x first order)

$$=\int_0^{\frac{\pi}{2}}\int_0^1\int_{r\cos\theta}^1 f\,r\,dx\,dr\,d\theta$$

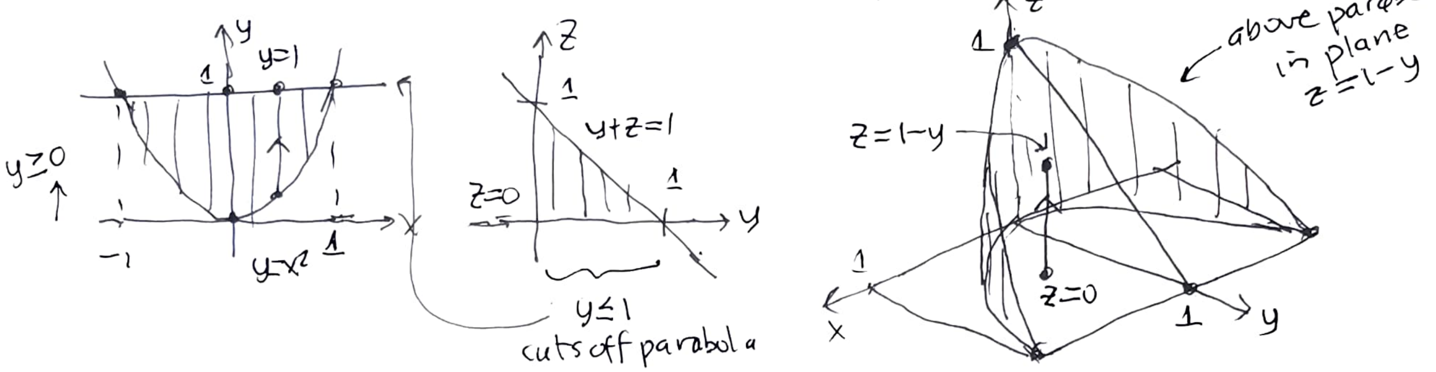
result: $\langle \bar{x},\bar{y},\bar{z}\rangle \approx \langle .45,.30,.46\rangle$

see Maple worksheet

values make sense

Center of Mass, Centroid (3)

Region R: enclosed by $y = x^2$, $z = 0$, $y + z = 1$
Find centroid and center of mass if density $\rho \propto y$. $(\rightarrow \rho = ky)$



reflection symmetry across vertical plane $x=0$ (above y axis)
of both region & density function ∴ must lie in $x=0$ plane.
Expect centroid to be closer to $y=0$ plane, but density pushes outward in y direction.

$z = 0..1-y$ while $y = x^2..1$ while $x = -1..1$

$$\iiint_R f\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f\,dz\,dy\,dx$$

"V" for volume moments!

centroid:

$$\langle V, V_{yz}, V_{xz}, V_{xy}\rangle = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} \langle 1, x, y, z\rangle\,dz\,dy\,dx$$

$$= \left\langle \frac{8}{15}, 0, \frac{8}{35}, \frac{16}{105}\right\rangle$$

$$\langle \bar{x}, \bar{y}, \bar{z}\rangle_c = \frac{\langle V_{yz}, V_{xz}, V_{xy}\rangle}{V} = \left\langle 0, \frac{3}{7}, \frac{2}{7}\right\rangle$$

$$\approx \langle 0, 0.43, 0.29\rangle$$

center of mass:

$$\langle M, M_{yz}, M_{xz}, M_{xy}\rangle = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} \langle 1, x, y, z\rangle\, ky\,dz\,dy\,dx$$

$$= k\left\langle \frac{8}{35}, 0, \frac{8}{63}, \frac{16}{315}\right\rangle$$

$$\langle \bar{x}, \bar{y}, \bar{z}\rangle = \frac{\langle M_{yz}, M_{xz}, M_{xy}\rangle}{M} = \left\langle 0, \frac{5}{9}, \frac{2}{9}\right\rangle$$

$$\approx \langle 0, \underset{0.43\nearrow}{0.56}, \underset{0.29\nearrow}{0.22}\rangle$$

centroid moves along y; down (because of wedge) makes sense.

(see Maple Worksheet)