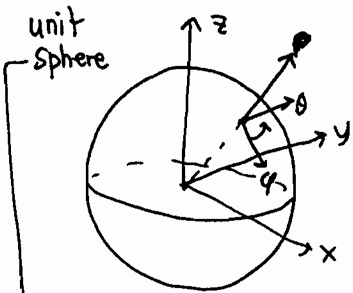
Surface integrals

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{S}} \vec{F}\cdot\hat{n}\, \underbrace{dS}_{\|\vec{N}\| dA} = \iint_{\text{uv region}} \vec{F}(\vec{r}(u,v))\cdot \vec{N}(u,v)\, dA$$

$\vec{N}$

$$\left.\begin{aligned} \vec{r} &= \vec{r}(u,v) \\ \vec{N} &= \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} \\ dA &= \text{"}du\,dv\text{"} \end{aligned}\right\}$$

unit sphere



$x^2+y^2+z^2=1$

outer normal

$z=\pm\sqrt{1-x^2-y^2}$

natural parametrization (ϕ,θ)

$\vec{r} = [\sin\phi\cos\theta, \sin\varphi\sin\theta, \cos\varphi]$

$\frac{\partial\vec{r}}{\partial\phi} = [-\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi]$

$\frac{\partial\vec{r}}{\partial\theta} = [-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0]$

$$\vec{N} = \begin{vmatrix} i & j & k \\ -C_\varphi C_\theta & C_\varphi S_\theta & -S_\varphi \\ -S_\varphi S_\theta & S_\varphi C_\theta & 0 \end{vmatrix} = \cdots = \sin\varphi\, \underbrace{\vec{r}}_{=\hat{r}}$$

function graph approach (x,y)

upper hemisphere

$\vec{r} = [x, y, \sqrt{1-x^2-y^2}]$

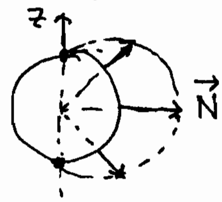
$\frac{\partial\vec{r}}{\partial x} = [1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}]$

$\frac{\partial\vec{r}}{\partial y} = [0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}]$

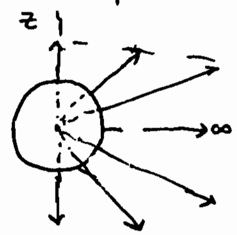
$$\vec{N} = \frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{-x}{\sqrt{1-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{1-x^2-y^2}} \end{vmatrix}$$

$= [+\frac{x}{\sqrt{1-x^2-y^2}}, \frac{+y}{\sqrt{1-x^2-y^2}}, 1]$

$= \frac{[x, y, \sqrt{1-x^2-y^2}]}{\sqrt{1-x^2-y^2}} \Big\} \leftarrow \vec{r}=\hat{r}$



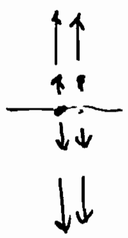
$\vec{N}$



lower hemisphere:

$\vec{N} = \frac{[x, y, -\sqrt{1-x^2-y^2}]}{\sqrt{1-x^2-y^2}}$

$\vec{F} = [0, 0, z]$



$\vec{F}(r(\varphi,\theta)) = [0, 0, \cos\varphi]$

$\vec{F}(r(\varphi,\theta))\cdot\vec{N}(\varphi,\theta) = \cos\varphi\,(\sin\varphi)(\cos\varphi)$

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\varphi\, d\varphi$$

$$= 2\pi\left(-\frac{\cos^3\varphi}{3}\right)\Big|_0^{\pi} = \frac{4\pi}{3}$$

upper:

$\vec{F}(\vec{r}(x,y)) = [0, 0, \sqrt{1-x^2-y^2}]$

$\vec{F}(\vec{r}(x,y))\cdot\vec{N}(x,y) = \sqrt{1-x^2-y^2}$

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{\underbrace{1-r^2}_{u}}\, \underbrace{r\,dr}_{-\frac{1}{2}du}\, d\theta$$

$$(2\pi)\left(-\frac{1}{2}\,\frac{2}{3}(1-r^2)^{3/2}\Big|_0^1\right) = \frac{2\pi}{3}$$

lower $\vec{F}(\vec{r}(x,y)) = [0, 0, -\sqrt{1-x^2-y^2}]$

$\vec{F}(\vec{r}(x,y))\cdot\vec{N}(x,y) = \sqrt{1-x^2-y^2}$

$\to$ same result, so double $\to$ $\frac{4\pi}{3}$ ✓

scalar integral over upper hemisphere:

$\iint_{\mathcal{S}} z\, dS \qquad dS = \|\vec{N}\|\, dA = \sin\varphi\, d\varphi\, d\theta = \frac{1}{\sqrt{1-x^2-y^2}}\, dy\, dx$

z: $\cos\varphi = \sqrt{1-x^2-y^2}$

$$= \int_0^{2\pi}\int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi\, d\theta \qquad = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,\frac{1}{\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$= 2\pi\, \frac{\sin^2\varphi}{2}\Big|_0^{\pi/2} = \pi \qquad = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx = \pi(1)^2$$

$$= \int_0^{2\pi}\int_0^1 r\, dr\, d\theta = 2\pi\, \frac{r^2}{2}\Big|_0^1 = \pi$$

$$\iint_{\mathcal{S}} dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi\, d\theta$$

$$= 2\pi\,(-\cos\varphi)\Big|_0^{\pi/2} = 2\pi$$

$$\iint_{\mathcal{S}} dS = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^1 \frac{r\, dr\, d\theta}{\sqrt{1-r^2}} = 2\pi(-1)(1-r^2)^{1/2}\Big|_0^1 = 2\pi$$

$$\bar{z} = \frac{\iint_{\mathcal{S}} z\, dS}{\iint_{\mathcal{S}} dS} = \frac{\pi}{2\pi} = \frac{1}{2}$$

center of gravity at $[0, 0, \frac{1}{2}]$

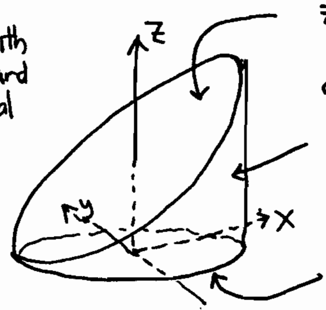
Note $\operatorname{div}\vec{F} = \frac{\partial z}{\partial z} = 1$

$$\iiint_{B} \operatorname{div}\vec{F}\, dV = V = \frac{4\pi}{3}(1)^3 = \frac{4\pi}{3}$$

B: ball inside sphere

surface integrals (2)

S with outward normal



S_3: $z = x+1 \to \vec r = [x, y, x+1] = [r\cos\theta, r\sin\theta, r\cos\theta+1]$ $\quad \begin{cases} r = 0..1 \\ \theta = 0..2\pi \end{cases}$

S_2: $x^2+y^2=1 \to \vec r = [\cos\theta, \sin\theta, z]$ $\quad \begin{cases} z = 0..\cos\theta+1 \\ \theta = 0..2\pi \end{cases}$

(cylindrical coordinate parametrizations)

S_1: $z=0$, $x^2+y^2 \le 1$ $\to \vec r = [r\cos\theta, r\sin\theta, 0]$ $\quad \begin{cases} r = 0..1 \\ \theta = 0..2\pi \end{cases}$

S_1: $\frac{\partial \vec r}{\partial r} = [\cos\theta, \sin\theta, 0]$ $\quad \frac{\partial \vec r}{\partial \theta} = [-r\sin\theta, r\cos\theta, 0]$ $\quad \frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta} = \begin{vmatrix} i & j & k \\ c & s & 0 \\ -rs & rc & 0 \end{vmatrix} = [0, 0, r(c^2+s^2)] = r[0,0,1]$

$\vec N = -r[0,0,1]$ outward normal.

S_2: $\frac{\partial \vec r}{\partial \theta} = [-\sin\theta, \cos\theta, 0]$ $\quad \frac{\partial \vec r}{\partial z} = [0,0,1]$ $\quad \frac{\partial \vec r}{\partial \theta} \times \frac{\partial \vec r}{\partial z} = \begin{vmatrix} i & j & k \\ -s & c & 0 \\ 0 & 0 & 1 \end{vmatrix} = [\cos\theta, \sin\theta, 0] = \vec N$

S_3: $\frac{\partial \vec r}{\partial r} = [c, s, c]$ $\quad \frac{\partial \vec r}{\partial \theta} = [-rs, rc, -rs]$ $\quad \frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta} = \begin{vmatrix} i & j & k \\ c & s & c \\ -rs & rc & -rs \end{vmatrix} = [-r(c^2+s^2), r(\cancel{cs-cs}), r(c^2+s^2)]$
$= r[-1, 0, 1] = \vec N$

$\vec F = [x, y, z+x^2] = \vec r + [0,0,x^2]$ position vector tilted up by an amount which increases with $|x|$.

$\therefore$ positive flux through S_2 and S_3 but negative flux through S_1 (i.e. up into solid).

oops!

$$\iint_{S_1} \vec F \cdot d\vec S = \int_0^{2\pi}\int_0^1 \underbrace{[r\cos\theta, r\sin\theta, (r\cos\theta)^2]}_{\vec F(\vec r(r,\theta))} \cdot \underbrace{[0,0,1](-r)}_{\vec N} \, dr\, d\theta$$

$$= -\int_0^{2\pi}\int_0^1 (\quad + r^3\cos^2\theta)\, dr\, d\theta = -\left[\quad \underbrace{\int_0^1 r^3 dr}_{\frac{r^4}{4}\Big|_0^1 = \frac14} \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{\int \frac{1-\cos 2\theta}{2} d\theta} \right]$$

$$\pi = \frac{\theta - \frac12 \sin 2\theta}{2}\Big|_0^{2\pi}$$

$$= -\left[\quad \frac{\pi}{4} \right] = -\frac{\pi}{4}$$

$$\iint_{S_2} \vec F \cdot d\vec S = \int_0^{2\pi}\int_0^{\cos\theta+1} \underbrace{[\cos\theta, \sin\theta, z+\cos^2\theta] \cdot [\cos\theta, \sin\theta, 0]}_{(\cos^2\theta+\sin^2\theta) + 0 = 1} \, dz\, d\theta = \int_0^{2\pi} (\cos\theta+1)\, d\theta = \sin\theta + \theta \Big|_0^{2\pi} = 2\pi$$

$$\iint_{S_3} \vec F \cdot d\vec S = \int_0^{2\pi}\int_0^1 \underbrace{[r\cos\theta, r\sin\theta, (r\cos\theta+1) + r^2\cos^2\theta] \cdot [-1,0,1]\, r}_{[-\cancel{r\cos\theta} + \cancel{r\cos\theta}+1+r^2\cos^2\theta]\, r \,=\, r + r^3\cos^2\theta} \, dr\, d\theta = \int_0^{2\pi} \Big(\underbrace{\frac{r^2}{2}\Big|_0^1}_{\frac12} + \underbrace{\frac{r^4}{4}\Big|_0^1}_{\frac14} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} \Big) d\theta$$

$$= \left\{ \frac12 \theta + \frac18 \left[\theta + \frac12 \sin 2\theta \right] \right\}_0^{2\pi} = \pi + \frac14 \pi + 0 = \frac{5\pi}{4}$$

so $\iint_S \vec F \cdot d\vec S = -\frac{\pi}{4} + 2\pi + \frac{5\pi}{4} = \boxed{3\pi}$

Note: $\operatorname{div} \vec F = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z + x^2 = 1+1+1 = 3$

$$\iiint_E \operatorname{div} F \, dV = 3 \iiint_E 1\, dV = 3 \int_0^{2\pi}\int_0^1 \underbrace{\int_0^{1+\cos\theta} 1\, r\, dz}_{r(1+\cos\theta)} \, dr\, d\theta = 3 \underbrace{\int_0^1 r\, dr}_{\frac12} \underbrace{\int_0^{2\pi} 1+\cos\theta \, d\theta}_{(\theta+\sin\theta)\big|_0^{2\pi} = 2\pi} = \boxed{3\pi}$$

E inside S