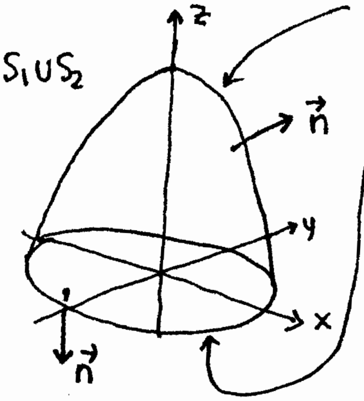
surface integrals

$S = S_1 \cup S_2$

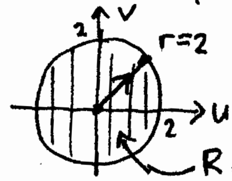


$S_1: \; z = 4 - x^2 - y^2, \; z \geq 0$ upward normal $\Big\} \; \vec{r}(u,v) = \langle u, v, 4-u^2-v^2 \rangle$

$S_2: \; z = 0, \; x^2+y^2 \leq 4$ downward normal $\Big\} \; \vec{r}(u,v) = \langle u, v, 0 \rangle$

$S = S_1 \cup S_2$ closed, outward normal

allowed parameter region:



$u = r\cos\theta$
$v = r\sin\theta$
$R: \; r \leq 2, \; 0 \leq \theta \leq 2\pi$

$\vec{F} = \langle xy, yz, \cancel{x}z \rangle$

$S_1: \; \vec{r}(u,v) = \langle u, v, 4-u^2-v^2 \rangle = \langle x, y, z \rangle$

$\vec{r}_u(u,v) = \langle 1, 0, -2u \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, -2v \rangle$

$\vec{n}(u,v) = \vec{r}_u(u,v) \times \vec{r}_v(u,v)$

$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle uv, v(4-u^2-v^2), u(4-u^2-v^2) \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v) = (2u)(uv) + (2v)(v)(4-u^2-v^2) + u(4-u^2-v^2)$

$= 2u^2v + (2v^2+u)(4-u^2-v^2)$

$S_2: \; \vec{r}(u,v) = \langle u, v, 0 \rangle = \langle x, y, z \rangle$

$\vec{r}_u(u,v) = \langle 1, 0, 0 \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, 0 \rangle$

$\vec{n}(u,v) = \vec{r}_u(u,v) \times \vec{r}_v(u,v)$

$= \hat{\imath} \times \hat{\jmath} = \hat{k} = \langle 0, 0, 1 \rangle \leftarrow$ (wrong direction: must change S_2 surface integral sign by hand)

$\vec{F}(\vec{r}(u,v)) = \langle uv, 0, 0 \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v) = 0$

$\iint_{S_2} \vec{F} \cdot d\vec{S} = 0$

$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_R 2u^2v + (2v^2+u)(4-u^2-v^2) \, dv\,du$ (use polar coords)

$= \int_0^{2\pi} \int_0^2 \left[2r^3\cos^2\theta\sin\theta + (2r^2\sin^2\theta + r\cos\theta)(4-r^2) \right] r\,dr\,d\theta$

$= \int_0^2 2r^3\,dr \int_0^{2\pi} \underbrace{\cos^2\theta}_{u^2} \underbrace{\sin\theta\,d\theta}_{-du} + \int_0^2 (4r^2-r^4)\,dr \underbrace{\int_0^{2\pi} \cos\theta\,d\theta}_{+\sin\theta\big|_0^{2\pi} = 0}$

$-\frac{u^3}{3} = -\frac{\cos^3\theta}{3}\Big|_0^{2\pi} = 0$

$+ \underbrace{\int_0^2 2(4r^3-r^5)\,dr}_{2\left(r^4 - \frac{r^6}{6}\right)\big|_0^2} \; \underbrace{\int_0^{2\pi} \sin^2\theta\,d\theta}_{\pi \text{ (MAPLE)}} = \boxed{\frac{32\pi}{3}} \rightarrow \iint_S \vec{F} \cdot d\vec{S}$

$= 2\left(16 - \frac{64}{6}\right) = \frac{32}{3}$

$= \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

$= \boxed{\frac{32\pi}{3}}$

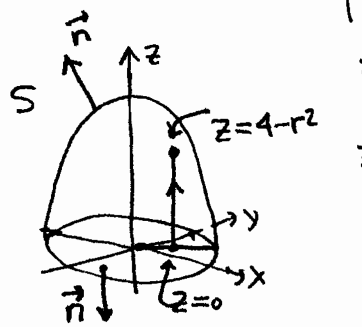
[Note Stewart 16.7.19 restricts S_1 to the square $0 \leq x \leq 1, \; 0 \leq y \leq 1$ yielding
$\int_0^1 \int_0^1 2u^2v + (2v^2+u)(4-u^2-v^2)\,dv\,du = \text{(MAPLE)} = \frac{713}{180}$]

surface integrals, Gauss's law, Stokes' theorem (2)

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(xy)+\frac{\partial}{\partial y}(yz)+\frac{\partial}{\partial z}(xz) = y+z+x = x+y+z$$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(xz)-\frac{\partial}{\partial z}(yz), \frac{\partial}{\partial z}(xy)-\frac{\partial}{\partial x}(xz), \frac{\partial}{\partial x}(yz)-\frac{\partial}{\partial y}(xy) \right\rangle$$

$$= \langle -y, -z, -x \rangle$$



S closed surface: top+bottom

interior E:

$z=0..4-r^2$

$r=0..2$

$\theta=0..2\pi$

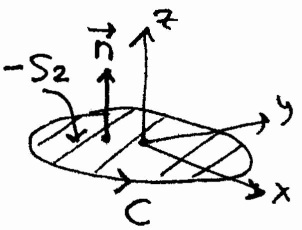
Gauss's law:

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV \quad \text{(use cyl coords)}$$

$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (\underbrace{r\cos\theta}_{\int_0^{2\pi}\cos\theta\,d\theta=0} + \underbrace{r\sin\theta}_{\int_0^{2\pi}\sin\theta\,d\theta=0} + z)\, r\,dz\,dr\,d\theta$$

$$= (2\pi)\int_0^2\int_0^{4-r^2} zr\,dz\,dr = 2\pi\int_0^2 \underbrace{\frac{z^2 r}{2}\Big|_{z=0}^{z=4-r^2}}_{\frac{r}{2}(4-r^2)^2 = \frac{r}{2}(16-8r^2+r^4) = 8r-4r^3+\frac{1}{2}r^5} dr$$

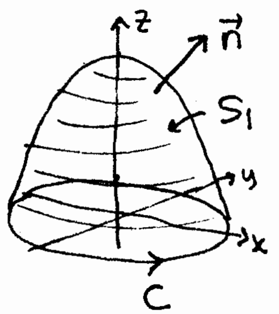
$$= 2\pi\int_0^2 8r-4r^3+\tfrac{1}{2}r^5\,dr = 2\pi\left(4r^2-r^4-\frac{r^6}{12}\right)\Big|_0^2 = 2\pi\left(16-16+\frac{64}{12}\right) = \boxed{\frac{32\pi}{3}} \checkmark$$



Stokes theorem

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-S_2} \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S}$$

upward normal



C: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$, $t=0..2\pi$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$\vec{F}(\vec{r}(t)) = \langle 4\cos t\sin t, 0, 0\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = -8\sin^2 t\cos t$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} -8\underbrace{\sin^2 t}_{u^2}\underbrace{\cos t\,dt}_{du} = -8\frac{\sin^3 t}{3}\Big|_0^{2\pi} = \boxed{0}$$

$$\iint_{-S_2} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_R \langle -v, 0, -u\rangle\cdot\underbrace{\langle 0,0,1\rangle}_{\text{upward normal}}\,dA = -\iint_R u\,dA = -\int_0^{2\pi}\int_0^2 (r\cos\theta)\,r\,dr\,d\theta = \boxed{0} \checkmark$$

$$\iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_R \underbrace{\langle -v, -(4-u^2-v^2), -u\rangle\cdot\langle 2u, 2v, 1\rangle}_{-2uv-2v(4-u^2-v^2)-u} dA$$

$$= \int_0^{2\pi}\int_0^2 \Big(-2r^3\underbrace{\cos\theta\sin\theta}_{-\frac{\cos 2\theta}{2}\big|_0^{2\pi}=0} \underbrace{-2r^2\sin\theta(4-r^2)}_{0} \underbrace{-r^2\cos\theta}_{0}\Big)\,dr\,d\theta = \boxed{0} \checkmark$$