deconstructing a triple integral & using rotational symmetry

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx$$

integration leads to more complication, stops Maple after 1st integration

sqrts are bad news for integrals ☹ Maple cannot evaluate this. What to do?

$$\int_{x=-2}^{x=2}\int_{y=x^2}^{y=4}\int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy$$

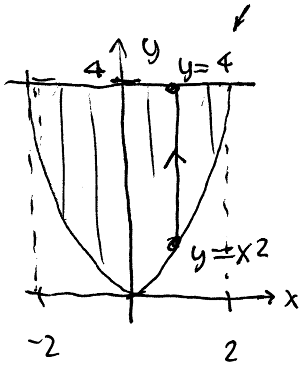
← annotate limits of integration to get equations of surfaces and lines bounding triple & double integration regions

easy 2-d diagram

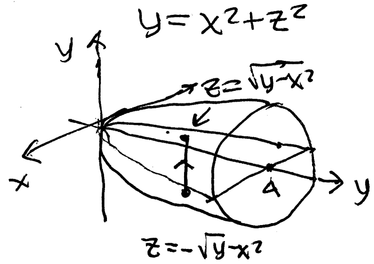
$z=\pm\sqrt{y-x^2}$ upper lower graphs

$z^2 = y-x^2$

$y = x^2+z^2$ parabola of revolution about y axis. cut off by plane $y=4$ from left diagram



outer double integral



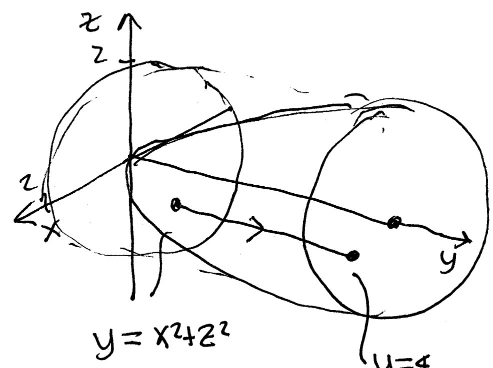
innermost integral

in fact $z=0$ cross-section is exactly the left diagram which separates upper & lower graphs

innermost integral goes from bottom to top

To take advantage of rotational symmetry we should integrate first in y direction.

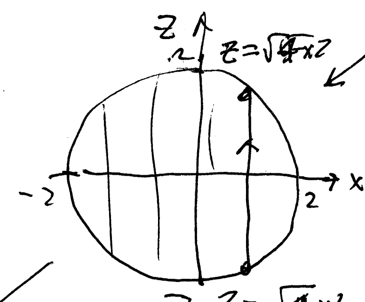
$y=x^2+z^2$ intersects $y=4$ at $x^2+z^2=4$ circle of radius 2 in xz plane (projection onto xz plane)



$y=x^2+z^2$

$y=4$

innermost integral



outer double integral

But Maple cannot do outer double integral in this order either (stops after 2nd integration)

If we keep Cartesian coords:

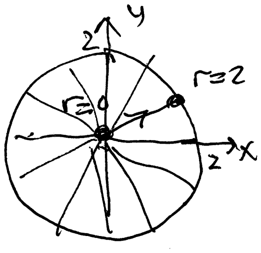
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx$$

dy(r dr dθ)

$$= \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r \; r\, dy\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2\, dy\, dr\, d\theta$$

$r^2 y\big|_{r^2}^{4} = r^2(4-r^2)$



$\theta = 0..2\pi$

$x=r\cos\theta$

$z=r\sin\theta$

$x^2+z^2=r^2$

introduce polar coords in xz plane

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{2} 4r^2-r^4\, dr}_{\frac{4r^3}{3}-\frac{r^5}{5}\Big|_0^2 = \frac{2^5}{3}-\frac{2^5}{5} = 2^5\left(\frac{2}{15}\right) = \frac{2^6}{15}} = \frac{2^7}{15}\pi = \boxed{\frac{128}{15}\pi}$$

3d diagram tells you starting & stopping values of innermost integral. projection onto coord plane gives outer double integral.