Decomposing a vector with respect to a direction

**Acute angle case**

\[ \vec{b} \cdot \hat{a} = |\vec{b}| \cos \theta \]

\[ \vec{b} = \vec{b}'' + \vec{b}^\perp \] (the vector)

vector component of \( \vec{b} \) perpendicular to \( \vec{a} \)

\[ \vec{b}^\|| = (\vec{b} \cdot \hat{a}) \hat{a} \] (the direction)

| alternative scenario |

| obtuse angle case |

only the direction of \( \hat{a} \) is relevant:

\[ \hat{a} = \frac{\vec{a}}{\|\vec{a}\|}, \quad |\hat{a}| = 1 \]

and this decomposes \( \vec{b} \) into the sum of two orthogonal (perpendicular) vectors

Note:

\[ |\vec{b}| \cos \theta = |\hat{a}| \|\vec{b}\| \cos \theta = \hat{a} \cdot \vec{b} = \text{scalar component of } \vec{b} \text{ along } \vec{a} \]

\[ |\vec{b}| \sin \theta = |\hat{a}| \|\vec{b}\| \sin \theta = |\hat{a} \times \vec{b}| = \text{scalar component of } \vec{b} \text{ perpendicular to } \vec{a} \]

**Example**

\[ \vec{b} = \langle 2,3,0 \rangle \]

results consistent with hand graph

\[ |\vec{a}| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10} \]

\[ \hat{a} = \frac{1}{\sqrt{10}} \langle 3,-1,0 \rangle \]

\[ \vec{b} \cdot \hat{a} = \frac{\langle 2,3,0 \rangle \cdot \langle 3,-1,0 \rangle}{\sqrt{10}} = \frac{6 - 3 + 0}{\sqrt{10}} = \frac{3}{\sqrt{10}} \]

\[ \vec{b}^\| = (\vec{b} \cdot \hat{a}) \hat{a} = \frac{3}{\sqrt{10}} \langle 3,-1,0 \rangle = \frac{3}{10} \langle 3,-1,0 \rangle = \langle \frac{9}{10}, \frac{-3}{10}, 0 \rangle \]

\[ \vec{b}^\perp = \vec{b} - \vec{b}'' = \langle 2,3,0 \rangle - \langle \frac{9}{10}, \frac{-3}{10}, 0 \rangle = \langle \frac{11}{10}, \frac{33}{10}, 0 \rangle \]

**Aside**

\[ |\vec{b}^2| = |\frac{11}{10} \langle 1,1,0 \rangle| = \frac{11}{10} \]

alternatively

\[ \hat{a} \times \vec{b} = \frac{1}{\sqrt{10}} \langle 1,1,0 \rangle \times \langle 3,-1,0 \rangle = \frac{1}{\sqrt{10}} \langle 0,0,9+2 \rangle = \frac{1}{\sqrt{10}} \langle 0,0,11 \rangle \]

\[ |\hat{a} \times \vec{b}| = \frac{1}{\sqrt{10}} \frac{11 \sqrt{10}}{10} = \frac{11}{10} \]
relative angle between 2 (nonzero) vectors

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = 0 \rightarrow |\mathbf{a}| = 0 \text{ or } |\mathbf{b}| = 0 \text{ or } \cos \theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} (90^\circ) \]

"orthogonal vectors"

have zero dot product

either \( \mathbf{a} \) or \( \mathbf{b} \) is zero (or both)

or both are nonzero so an angle is defined and they are "perpendicular" vectors

Remark: \( |\mathbf{a}| = 0 \) is equivalent to \( \mathbf{a} = 0 \)

\( \sqrt{a_1^2 + a_2^2 + a_3^2} \rightarrow a_1 = a_2 = a_3 \rightarrow \mathbf{a} = 0 \Rightarrow \mathbf{a} = (0, 0, 0) \)

only the zero vector has zero length (not true in the geometry of the Euclidean plane used in special relativity!!)

"parallel vectors"

are proportional: \( \mathbf{a} \propto \mathbf{b} \Leftrightarrow \mathbf{a} = c \mathbf{b} \)

How to tell?

example: \( \langle 1, 2, 3 \rangle, \langle 2, 4, 6 \rangle \)

\( 2 \mathbf{a} = \mathbf{b} \Rightarrow \mathbf{a} = \frac{1}{2} \mathbf{b} \)

2 = c \rightarrow c = 2 \checkmark

4 = 2c \rightarrow c = 2 \checkmark

8 = 4c \rightarrow c = 2 \checkmark

c = 2

c consistent solution.

or ratios of corresponding components same:

\( \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = 2 = c \checkmark \)

projection processes

resolving a vector along 2 other vectors is a projection process (oblique projection)

with only 2 vectors, we can project one along the other using an orthogonal (= perpendicular) projection process, resolving one vector into a piece along the other (parallel to!) and a piece perpendicular to the other

\[ \mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_\perp \]

\( \mathbf{a}_{\parallel} \)

is orthogonal to \( \mathbf{a} \)

\( \mathbf{a}_{\parallel} \) along \( \mathbf{a} \)

\( \mathbf{a}_\perp \) along \( \mathbf{b} \)

\( \mathbf{a} \) along \( \mathbf{a} \)

(might be zero!!)