

multivariable "antidifferentiation" (potential for conservative vector field)

one independent variable

$$F(t) = \frac{df(t)}{dt} \rightarrow f(t) - f(0) = \int_0^t F(u) du$$

given  $\frac{df(t)}{dt}$  to find  $f(t)$   
 variable upper limit  $\int_0^t$   
 $\frac{df(u)}{du}$

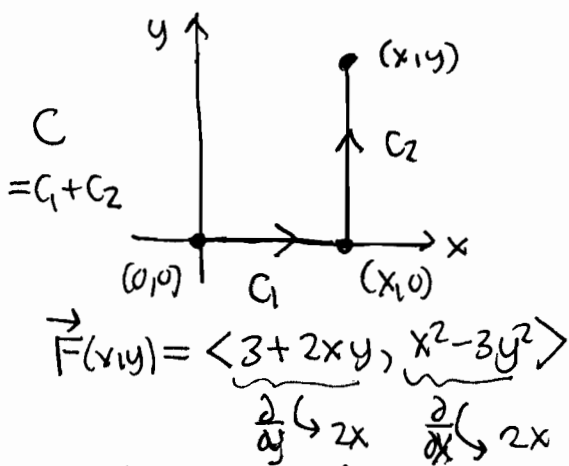
is an integral formula for an antiderivative  $f$  modulo the arbitrary constant value  $f(0) = C$

multivariable analog (2D)

$$\vec{F}(x,y) = \vec{\nabla} f(x,y) \rightarrow f(x,y) - f(0,0) = \int_C \vec{F}(x,y) \cdot d\vec{r}$$

given  $\vec{F}(x,y)$  to find  $f(x,y)$   
 new called potential function  
 if it exists then it doesn't matter what curve we use to get from our reference point  $[(0,0)]$  for simplicity to a general point  $(x,y)$   
 variable endpoint  $C: (0,0) \rightarrow (x,y)$   
 is any curve

so we can use two successive partial integrations moving along one coordinate line and then another:



$C_1: \vec{r}(t) = \langle t, 0 \rangle \quad \vec{r}'(t) = \langle 1, 0 \rangle \quad t = 0 \dots x$

$\vec{F}(\vec{r}(t)) = \langle 3, x^2 \rangle \quad \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 3$   
 $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^x 3 dt = 3t \Big|_0^x = \boxed{3x}$

$C_2: \vec{r}(t) = \langle x, t \rangle \quad \vec{r}'(t) = \langle 0, 1 \rangle \quad t = 0 \dots y$

$\vec{F}(\vec{r}(t)) = \langle 3 + 2xt, x^2 - 3t^2 \rangle \quad \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = x^2 - 3t^2$   
 $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^y x^2 - 3t^2 dt = x^2t - t^3 \Big|_0^y = \boxed{x^2y - y^3}$

$\int_C \vec{F} \cdot d\vec{r} = 3x + x^2y - y^2$  so  
 $f(x,y) = \frac{f(0,0)}{C} + 3x + x^2y - y^2$

PDE approach

$$\frac{\partial f}{\partial x} = F_1(x,y)$$

$$\frac{\partial f}{\partial y} = F_2(x,y)$$

is a system of Partial Differential Equations for an unknown function  $f(x,y)$  but if it exists, its mixed 2nd partials must agree:

$$\left[ \begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} F_1 \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} F_2 \end{aligned} \right] \text{ must be equal } \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

called an integrability condition for the PDE system (condition for being able to "integrate" the system, i.e., solve it by integration techniques).

An equivalent partial integration technique can be used to solve one of these two equations & then backsub into the second to require it to be satisfied:

$$\int \left[ \frac{\partial f}{\partial x} = 3 + 2xy \right] dx \rightarrow f = \int 3 + 2xy dx = 3x + x^2y + C(y)$$

← "constant" may depend on  $y$  held fixed

$$\frac{\partial f}{\partial y} = x^2 - 3y^2 \rightarrow \frac{\partial}{\partial y} (3x + x^2y + C(y)) = x^2 - 3y^2$$

$x^2 + \frac{dC(y)}{dy} = x^2 - 3y^2 \rightarrow \frac{dC(y)}{dy} = -3y^2 \rightarrow C(y) = -y^3 + k$

so  $f = 3x + x^2y - y^3 + k$  same result.