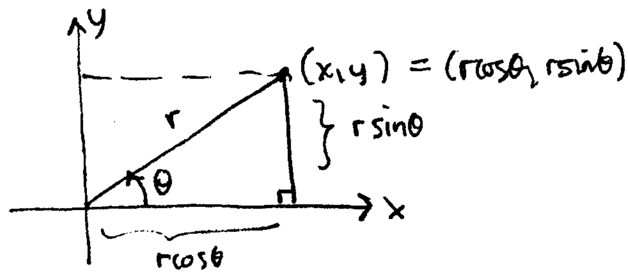
partial derivatives and changing coordinate systems

$$x = r\cos\theta \quad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$



$$y = r\sin\theta \quad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial r}{\partial \theta} = r\cos\theta$$

$$z = f(x,y) = f(x(r,\theta), y(r,\theta))$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y} \rightarrow \frac{1}{r}\frac{\partial z}{\partial \theta} = -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \cos^2\theta \left(\frac{\partial z}{\partial x}\right)^2 + 2\cos\theta\sin\theta \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \sin^2\theta \left(\frac{\partial z}{\partial y}\right)^2$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \sin^2\theta \left(\frac{\partial z}{\partial x}\right)^2 - 2\cos\theta\sin\theta \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \cos^2\theta \left(\frac{\partial z}{\partial y}\right)^2$$

$$\boxed{\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2} = \underbrace{(\cos^2\theta + \sin^2\theta)}_{1}\left(\frac{\partial z}{\partial x}\right)^2 + 0\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\left(\frac{\partial z}{\partial y}\right)^2$$

$$\boxed{= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \quad \checkmark$$

notation: $\vec{\nabla} z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y}\right) z$ (gradient of z)

then $|\vec{\nabla} z|^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$ (gradient squared of z)

this formula then shows how to evaluate the magnitude $|\vec{\nabla} z|$ of the gradient of a function in polar coordinates