Derivatives of 2D & 3D functions

gradient vector
\[ f(x,y) \rightarrow \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \]
\[ f(x,y,z) \rightarrow \nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \]
\[ \nabla f(P), \quad P = \{ x_1, y_1 \} \]

directional derivative of \( f \) in the direction \( \vec{u} \) at \( P \):
\[ D_{\vec{u}} f(P) = \vec{u} \cdot \nabla f(P) \]

2-D level curve
\[ \vec{\nabla} f(P_0) = \vec{n} \]
\[ f(P_0) = c \]

3-D level surface
\[ \vec{\nabla} f(x_0, y_0, z_0) = \vec{n} \]
\[ f(P_0) = c \]

normal line to level curve/surface:
thru \( P_0 \) along \( \vec{n} = \vec{\nabla} f(P_0) \)
2D \[ x = x_0 + t n_1 \]
\[ y = y_0 + t n_2 \]
3D \[ x = x_0 + t n_1 \]
\[ y = y_0 + t n_2 \]
\[ z = z_0 + t n_3 \]

2-D max-min:
find critical pts \( f_x(x_0, y_0) = 0 \) \( \neq f_y(x_0, y_0) \), use 2nd der. test
\[ f_{xx}(x_0, y_0) = 0 \]
\[ f_{xx}(x_0, y_0) < 0 \]
if also \( x_0 \) constant cross-section, consistent:
\[ f_y(x_0, y_0) = 0 \]
\[ f_{yy}(x_0, y_0) < 0 \]

appears to be local max:
\[ f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 \]
if \( > 0 \) confirma local max in all directions
\[ = 0 \] inconclusive
\[ < 0 \] saddle, no local extremum

similarly both positive... local min if confirmed
Derivatives of 2D & 3D functions (2)

tangent plane to graph of 2D function in 3D

\[ z = f(x_0, y_0) \]

\[ \text{3-vector} \]

\[ \mathbf{N} \]

via linear approx:

\[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

\[ f(x_0, y_0) \]

\[ L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

\[ L(9, 2, 1) = f(1, 2) + f_x(1, 2)(9 - 1) + f_y(1, 2)(2, 1 - 2) \]

linear approximations:

\[ f(x) \approx f(r_0) \quad \text{ref. pt. value} \]

\[ \frac{df(r_0)}{d\mathbf{r}} \quad \text{increment} \]

\[ \{ \text{differential of } f \} \quad \text{at } r_0 \]

2-D

\[ L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

\[ L(9, 2, 1) = f(1, 2) + f_x(1, 2)(9 - 1) + f_y(1, 2)(2, 1 - 2) \]

3-D

\[ L(x_0, y_0, z_0) \]

\[ L(9, 2, 1, 0) = f(1, 2, 0) + f_x(1, 2, 0)(9 - 1) + f_y(1, 2, 0)(2, 1 - 2) + f_z(1, 2, 0)(1 - 0) \]

\[ \text{Note: } L(x, y, z) = f(x_0, y_0, z_0) \text{ simplifies to tangent plane to level surface} \]

\[ f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0 \]

\[ \text{or } \nabla f(r_0) \cdot (r - r_0) = 0 \quad \text{gradient is orthogonal to increment vector away from point of tangency} \]

2D & 3D:

\[ F(x, y, z) = z - f(x, y) = 0 \]

\[ \text{graph of } f \text{ is level surface of } F, \text{ gradient of } F \text{ is normal to tangent plane to } f = \text{ tangent plane to level surface of } F \]

\[ \mathbf{N} = \nabla F(x_0, y_0, z_0) = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle \]

\[ \mathbf{N} \cdot (r - r_0) = 0 \]

same result as