Derivatives of 2D & 3D functions

gradient vector

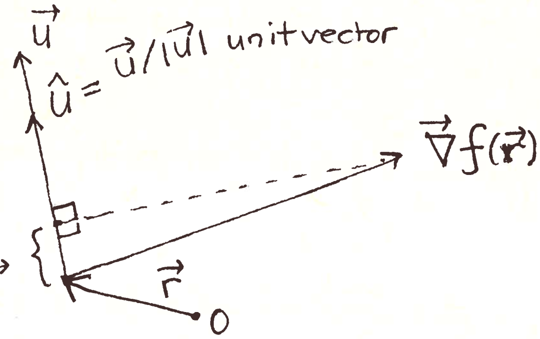
$f(x,y) \rightarrow \vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$f(x,y,z) \rightarrow \vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

$\Big\} \; \vec{\nabla} f(\vec{r}), \quad \vec{r} = \begin{cases} \langle x,y \rangle \\ \langle x,y,z \rangle \end{cases}$

directional derivative of f in the direction $\hat{u}$ at $\vec{r}$:

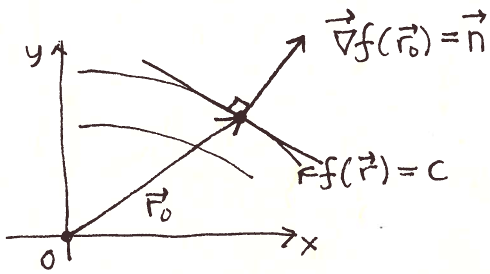
$$D_{\hat{u}} f(\vec{r}) = \hat{u} \cdot \vec{\nabla} f(\vec{r})$$



length $|\vec{\nabla} f(\vec{r})|$ = max rate of change at $\vec{r}$

direction $\widehat{\vec{\nabla} f(\vec{r})} = \dfrac{\vec{\nabla} f(\vec{r})}{|\vec{\nabla} f(\vec{r})|}$ = direction of max rate of change (when nonzero)

2-D level curve



tangent line $\perp$ gradient: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

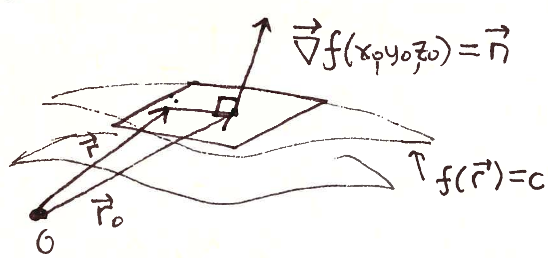
$\langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0$

$f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) = 0$

$\rightarrow$ solve for y: $y = y_0 - \dfrac{f_x(x_0,y_0)}{f_y(x_0,y_0)}(x - x_0)$

eqn of tangent line to level curve

3-D level surface



tangent plane $\perp$ gradient: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$\langle f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0) \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0) = 0$

eqn of tangent plane to level curve

normal line to level curve/surface: thru $\vec{r}_0$ along $\vec{n} = \vec{\nabla} f(\vec{r}_0)$

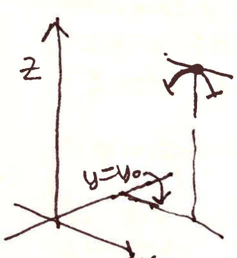
$\vec{r} = \vec{r}_0 + t\vec{n}$

2D $\begin{aligned} x &= x_0 + t n_1 \\ y &= y_0 + t n_2 \end{aligned}$

3D $\begin{aligned} x &= x_0 + t n_1 \\ y &= y_0 + t n_2 \\ z &= z_0 + t n_3 \end{aligned}$

2-D max-min: find critical pts $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$, use 2nd der. test

y-constant cross-section local max:



$\left.\begin{aligned} f_x(x_0,y_0) &= 0 \\ f_{xx}(x_0,y_0) &< 0 \end{aligned}\right\}$

if also x=constant cross-section consistent:

$\left.\begin{aligned} f_y(x_0,y_0) &= 0 \\ f_{yy}(x_0,y_0) &< 0 \end{aligned}\right\}$

appears to be local max

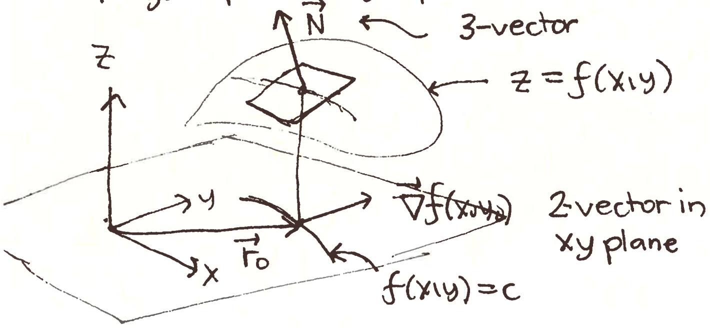
$f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$

- > 0 confirms local max in all directions
- $= 0$ inconclusive
- < 0 saddle, no local extremum

similarly both positive ... local min if confirmed

Derivatives of 2D & 3D functions (2)

tangent plane to graph of 2D function in 3D



via linear approx:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$z_0 = f(x_0,y_0)$

$$\left(z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = L(x,y) \right)$$

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0$$

$$\vec{N} = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle$$

$2D \cap 3D$:

$$F(x,y,z) = z - f(x,y) = 0$$

graph of f is level surface of F, gradient of F is normal to tangent plane to f = tangent plane to level surface of F

$$\vec{N} = \vec{\nabla} F(x_0,y_0,z_0) = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle$$

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0$$

same result as

ind. variable differentials

$$d\vec{r} = \vec{r} - \vec{r}_0 = \begin{cases} \langle dx, dy \rangle \\ \langle dx, dy, dz \rangle \end{cases}$$

linear approximations:

$$f(\vec{r}) \approx \underbrace{f(\vec{r}_0)}_{\text{ref. pt. value}} + \underbrace{(\vec{r}-\vec{r}_0) \cdot \vec{\nabla} f(\vec{r}_0)}_{df(\vec{r}_0) \text{ increment}} = L(\vec{r})$$

$df(\vec{r}_0)$ } differential of f at $\vec{r}_0$

2-D

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L(.9,2.1) = f(1,2) + \underbrace{f_x(1,2)\underbrace{(.9-1)}_{dx} + f_y(1,2)\underbrace{(2.1-2)}_{dy}}_{df}$$

3-D

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$L(.9,2.1,.1) = f(1,2,0) + \underbrace{f_x(1,2,0)\underbrace{(.9-1)}_{dx} + f_y(1,2,0)\underbrace{(2.1-2)}_{dy} + f_z(1,2,0)\underbrace{(.1-0)}_{dz}}_{df}$$

[Note: $L(x,y,z) = f(x_0,y_0,z_0)$ simplifies to tangent plane to level surface

$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$

or $\vec{\nabla} f(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) = 0$ gradient is orthogonal to increment vector away from point of tangency]