① multivariable integration: visualizing the iteration process

background: calc 2 integration

example:

Suppose we have an explicit functional relationship like: $y = f(x,p) = x^p$

y ↑ dependent variable; x ↑ independent variable; p ↑ parameter

We have no problem integrating this function

$$\int_1^2 x^p\,dx \overset{p\neq -1}{=} \underbrace{\frac{x^{p+1}}{p+1}}_{\text{antiderivative of } x^p}\Bigg|_1^2 = \frac{2^{p+1}-1}{p+1}.$$

The interval of integration is from $x=1$ to $x=2$ or for short $x=1..2$

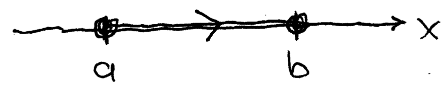
In general $\int_a^b f(x,p)\,dx \geq 0$ if $f(x,p)\geq 0$ and $a\leq b$

We assume initially the "lower limit" a is less than or equal to the upper limit to draw this conclusion, but then we extend the notation

$$\int_2^1 f(x,p)\,dx = -\int_1^2 f(x,p)\,dx$$

The limits of integration must always be ordered $a\leq b$ to draw sign conclusions from the sign of the integrand.

We can illustrate $\int_a^b \ldots dx$ where $a<b$ by



We "integrate up" the values of $f(x,p)$ as x increases from a to b, like a store scanner scanning from the left endpoint to the right endpoint.

Calc 3 integration

Multivariable integration is all about describing the region of integration and translating that into limits of integration of an "iterated" integral, namely repeated nested calc 2 integrals each successively evaluated using antiderivatives. This is done by parametrizing regions of the plane or space (or higher dimensions) using bounding curves and surfaces which limit the ranges of the independent variables. These curves or surfaces are specified by direct functional relationships which specify one independent variable in terms of one or more of the remaining variables, say $y=g(x)$ in the plane or $z=h(x,y)$ in space in Cartesian coordinates.

When we set up nested iterated integrals, interpreted as integrals over a region of the plane/space, (the space of the multiple independent variables), it is understood that the limits of integration have to be ordered:
"lower limit" $\leq$ "upper limit" (We can later free to exchange them and change the sign of the integral.)

② multivariable integration

Suppose $a \le b$, $c \le d$. A nested integral is defined by:

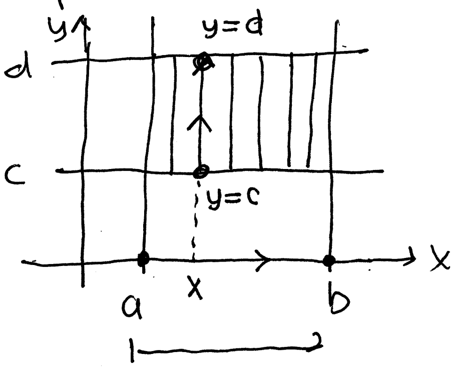
$$\int_a^b \int_c^d f(x,y)\,dy\,dx \equiv \int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy} \Big)\,dx$$

"partially integrate" wrt y from c to d holding x fixed as though it were a parameter; the result is a function of x only

↳ next integrate the resulting function of x from a to b to get a number

This is still basically just calc 2 integration, done twice successively like taking a second derivative by differentiating a derivative.

Multivariable calculus enters the story in its interpretation. For each value $x = a..b$, we consider, we integrate over a directed line segment in the y direction from c to d (think of scanning up the values of $f(x,y)$ along this line segment). As we sweep x from a to b, the typical vertical cross-section representing the y integration sweeps across the rectangle pictured here.



cross-section sweeps across this interval on the x-axis

By drawing equally spaced vertical linear cross-sections of the rectangle, and labeling a typical one at its bullet endpoints by the starting and stopping values of y as equations, with an arrowhead midway to indicate the direction of increase for the integration variable y

This is really a scheme to parametrize all the points (x,y) in this rectangle:
$y = c..d$ while $x = a..b$

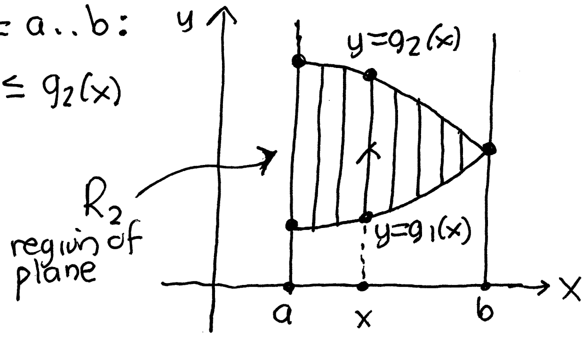
Viceversa this diagram/parametrization directly translates back to the "iterated" double integral. The labels on the vertical cross-sections directed along the lines of increasing y (while x is fixed) translate directly to the lower & upper limits of integration on the inner integral, while the endpoints of the interval of x values integrated over become the outer limits of integration (left to right for x, bottom to top for y).

Note that we can easily exchange x and y in this discussion and talk about horizontal linear cross-sections.

(3) multivariable integration

We can easily generalize the previous diagram to illustrate integrating first in the y direction (at fixed x) between the graphs of two functions of x which are always ordered so that one is always less than or equal to the other over the interval of x values considered.

on $x = a..b$: $g_1(x) \le g_2(x)$



This represents a parametrization of all the points in this "shaded" region

$y = g_1(x) .. g_2(x)$ while $x = a..b$

$\longrightarrow$ increase bottom to top

$\longrightarrow$ increase left to right

The corresponding integral then has an obvious interpretation.

$$\int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy}\, dx$$

$$= F(x,y)\Big|_{y=g_1(x)}^{y=g_2(x)}$$

$$= \underbrace{F(x,g_2(x)) - F(x,g_1(x))}_{\text{function only of } x}$$

$$= \int_a^b (\ )\, dx$$

Find $F(x,y)$ such that

$\underbrace{\frac{\partial F}{\partial y}}(x,y) = f(x,y)$ (antiderivative wrt y holding x fixed)

"partial derivative" holding x fixed as though just a parameter

That the inner limits of integration depend on x makes no difference because x is held fixed during this operation wrt y.

We are just "integrating up" all the values of $f(x,y)$ over the region R_2 of the plane.

The outer limits of integration give the endpoints of the interval on the x-axis obtained by projecting all the points down to the x-axis along the y-direction (vertical projection down to horizontal axis). In the Cartesian coordinate grid, the y-direction (holding x fixed) is vertical, increasing from bottom to top, while the x-direction (holding y fixed) is horizontal, increasing from left to right.

Multivariable calculus explains why this integral equals the limiting Riemann sum expression which defines the integral of the function f over the region R_2 with respect to the differential of volume:

$$\iint_{R_2} f\, dV$$

④ multivariable integration

We can continue this scheme to 3 dimensions in Cartesian coords (x, y, z).

First nesting:

$$\int_a^b \int_c^d \int_m^n f(x,y,z)\, dz\, dy\, dx = \int_a^b \left(\int_c^d \left(\int_m^n f(x,y,z)\, dz \right) dy \right) dx$$

integrate over z – result is function only of (x,y)

next integrate over y – result is a function only of x

finally integrate over x – result is a number

We can generalize it in the same way:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

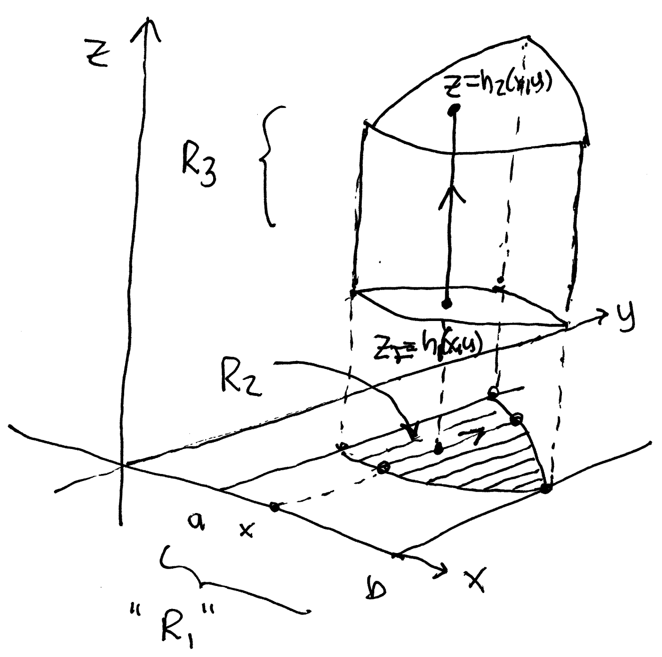
(x,y) are parameters held fixed, integrate wrt z to get a function only of (x,y)

x is a parameter, integrate wrt y to get a function only of x

Finally integrate over x to get a number

Assuming $h_1(x,y) \le h_2(x,y)$ while $y = g_1(x) \dots g_2(x)$ and $g_1(x) \le g_2(x)$ while $x = a..b$, then we get a parametrized region R_3 of space:

$z = h_1(x,y) .. h_2(x,y)$
while $y = g_1(x) .. g_2(x)$
while $x = a..b$.



Above each point in R_2, we integrate from $z = h_1(x,y)$ upward to $z = h_2(x,y)$ represented by the bullet endpoint linear cross-section in the z-direction

We then integrate from $y = g_1(x)$ to $y = g_2(x)$ for each x in the interval

First we project to the x-y plane, then to the x-axis.

This diagram with its successive labeled typical linear cross-sections directly translates into the corresponding limits of integration of the iterated triple integral.

We can continue this process into higher dimensions (but rarely in ak3!). We can also permute the coordinates (x,y,z) in this discussion.

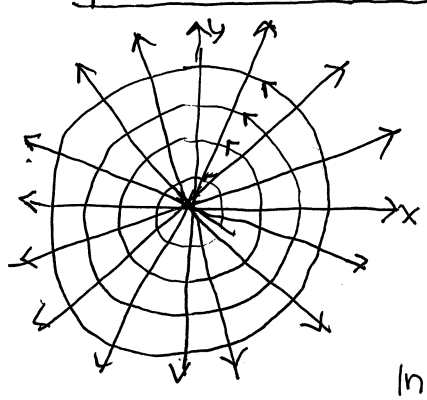
⑤ multivariable integration : other coordinate systems

This process is intimately connected to the Cartesian coordinate grid in the plane or space since our individual partial integrations are aligned with straight line segments which are part of that grid.

We can easily adapt the same process to non-Cartesian coordinate systems like polar coordinates in the plane and cylindrical and spherical coordinates in space.

Each partial integration follows a coordinate line (holding the remaining coordinates fixed). The peeling off of the integrals as we work from the innermost to the most outer integral corresponds to projecting along the coordinate lines.

polar coordinates(r,θ) r measures distance from origin: $r \geq 0$ always for integration purposes



the radial coordinate lines (increasing r) are half rays from the origin (constant angle θ).

θ measures the angle from the positive x-axis in the counter-clockwise direction.

the angular coordinate lines (constant r) are concentric circles about the origin.

In practice it is useful to specify curves only in the form $r = g(\theta)$ so we continue thinking of r like y and θ like x in the plane discussion, and we will therefore move first in the radial direction & then sweep around the angular direction.

There is an additional complication — we want our iterated integrals to represent an abstract $\iint_{R_2} f(r,\theta)\, dA$ integral with respect to the differential of area in the plane.

$dA = \underbrace{(r\,d\theta)(dr)}$

limiting differential of area = product of arclength differentials along orthogonal coordinate lines (explained in calc 3).

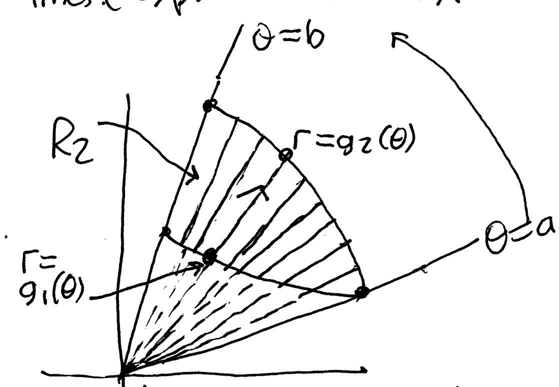
take original function of (x,y) & re-express in terms of (r,θ) via:

$x = r\cos\theta$
$y = r\sin\theta$.

Iterated integrals, r first, then θ:

$$\underbrace{\int_a^b \underbrace{\int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr}_{\text{result is function of } \theta}\, d\theta}_{\text{result is a number}}$$

(assuming $g_1(\theta) \leq g_2(\theta)$ while $\theta = a..b$)



Tall radial cross-sections converge at origin if extended

R_2 shaded by equally spaced (in angle) radial cross-sections:

inner integral outward in radial direction,

outer integral sweeps radial cross-section through region.

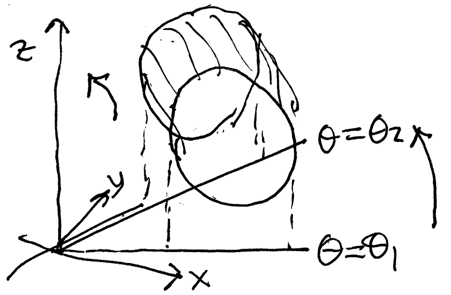
Note $a < b$! ordered!

(c) multivariable integration

Thus in polar coordinates double integrals correspond to parametrizations:
$r = g_1(\theta)\,..\,g_2(\theta)$ while $\theta = a\,..\,b$.

cylindrical and spherical coordinates in space

cylindrical coordinates (r,θ,z) replace (x,y) in cartesian coordinates (x,y,z) by their corresponding polar coordinates. For solid regions which are a wedge of a solid of revolution about the z-axis, bounded by two half planes $\theta=\theta_1$ and $\theta=\theta_2$ with $\theta_1<\theta_2$, a triple integral is most naturally formulated as an internal double integral in the r-z half plane (half since $r\geq 0$ for all integration matters) followed by an outer integral over $\theta=\theta_1\,..\,\theta_2$. An entire solid of revolution would correspond to $\theta = 0\,..\,2\pi$ or $\theta=-\pi\,..\,\pi$.



Since the solid region plane cross-section by the r-z halfplanes of fixed θ is independent of θ, the inner double integral of a triple integral is equivalent to the previous discussion in the x-y plane, relabling x as r and y as z (apart from the restriction $r\geq 0$).

Two integration orders are possible, horizontal and vertical first. Introducing polar coordinates $r=\rho\sin\phi$, $z=\rho\cos\phi$ with $\rho\geq 0$, $\phi=0\,..\,\pi$ measuring the new angle down from the positive z-axis corresponds exactly to the previous polar coordinate discussion apart from switching horizontal and vertical for the definition of the angle ϕ.

The differential of volume is:

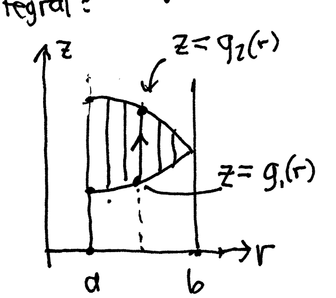
$$dV = dr\,dz\,(r\,d\theta) = ((d\rho)(\rho\,d\phi))(r\,d\theta) = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \text{ (radial)}$$
$$= r\,dz\,dr\,d\theta \text{ (vertical)}$$
$$= r\,dr\,dz\,d\theta \text{ (horizontal)}$$

↑ innermost integration direction in r-z plane.

Thus we have three types of parametrizations for triple integrals in these coordinates.

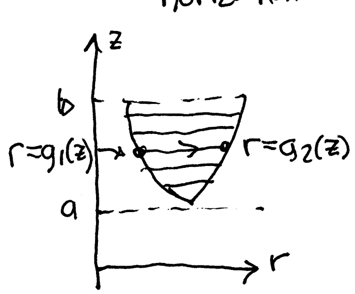
innermost integral:

vertical:



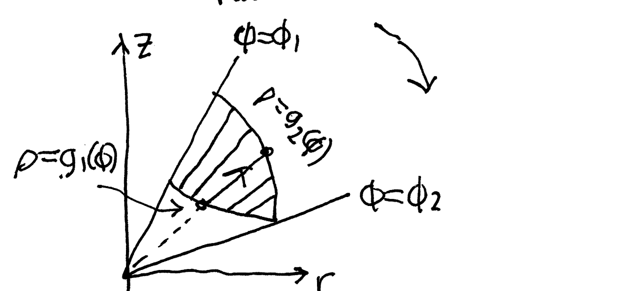
$$\int_{\theta_1}^{\theta_2}\int_a^b\int_{g_1(r)}^{g_2(r)} f(r\theta z)\,r\,dz\,dr\,d\theta$$

horizontal:



$$\int_{\theta_1}^{\theta_2}\int_a^b\int_{g_1(z)}^{g_2(z)} f(r\theta z)\,r\,dr\,dz\,d\theta$$

radial:



$$\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{g_1(\phi)}^{g_2(\phi)} f(\rho\phi\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$