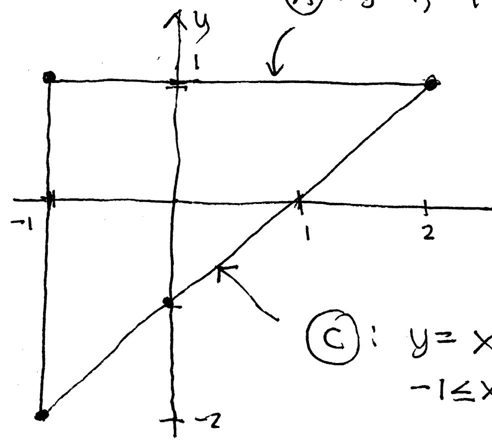
Max-Min with boundary example

extremize $f(x,y) = x^2+2xy+3y^2$
on the triangle with vertices
$(2,1)$, $(-1,1)$, $(-1,-2)$.



(A): $y=1$, $-1 \le x \le 2$

(B) $x=-1$
$-2 \le y \le 1$

(C): $y = x-1$ (slope intercept!)
$-1 \le x \le 2$

interior critical points:

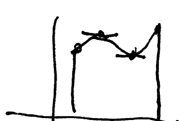
$f_x = 2x+2y = 2(x+y) = 0 \to y=-x \longrightarrow y=0$

$f_y = 2x+6y = 2(x+3y) = 0 \to x-3x=0 \to x=0$

$(0,0)$ is only critical point.
$f(0,0) = 0$

$\left.\begin{array}{l} f_{xx} = 2 > 0 \checkmark \\ f_{yy} = 6 > 0 \checkmark \end{array}\right\}$ local min?

$f_{xy} = 2$ —— $f_{xx}f_{yy} - f_{xy}^2 = 2(6) - 2^2 > 0$ yes.

calc I : 

continuous function on a closed interval must have global max and min either at critical points in interior or at endpoints.

so we have 3 calc I max min problems to find local extrema on boundary. Then we pick the least local min and the greatest local max from all these points.

(A) $y=1$: $-1 \le x \le 2$

$f(x,1) = x^2+2x+3 = g(x)$, $0 = g'(x) = 2x+2 \to x=-1$ also endpoint. local min at $(-1,1)$ (concave up parabola)

$f(-1,1) = 1-2+3 = 2$

(B) $x=-1$: $-2 \le y \le 1$

$f(-1,y) = 1-2y+3y^2 = h(y)$, $0 = h'(y) = 6y-2 \to y = 1/3$ local min at $(-1, 1/3)$ (concave up parabola)

$f(-1, 1/3) = 1 - \frac{2}{3} + 3\left(\frac{1}{9}\right) = \frac{2}{3}$

(C) $y=x-1$: $-1 \le x \le 2$

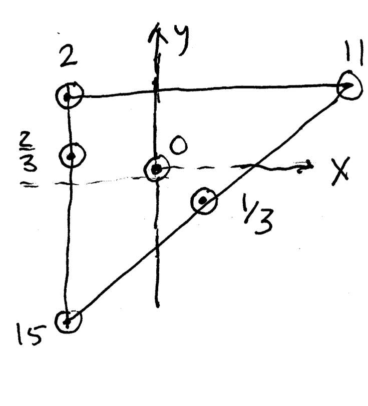
$f(x,x-1) = x^2 + 2(x-1)x + 3(x-1)^2$
$= x^2+2x^2-2x+3x^2-6x+3$
$= 6x^2-8x+3 = j(x)$

$0 = j'(x) = 12x-8 \to x = 2/3 \to y = \frac{2}{3}-1 = -\frac{1}{3}$

local min at $\left(\frac{2}{3}, -\frac{1}{3}\right)$

$f\left(\frac{2}{3},-\frac{1}{3}\right) = \frac{4}{9} - \frac{4}{9} + 3\left(\frac{1}{9}\right) = \frac{1}{3}$

remaining endpoints:
$f(2,1) = 4+4+3 = 11$
$f(-1,-2) = 1+2+12 = 15$



global min 0 at $(0,0)$
global max 15 at $(-1,-2)$.