the tangent plane and the linear approximation and differentials

\[ Z = f(x, y) \]

approximated at \( (x_0, y_0) \) by tangent plane

\[ Z = L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

where \( f_x(x_0, y_0) \) and \( f_y(x_0, y_0) \) are the partial derivatives of \( f \) at \( (x_0, y_0) \).

Increment \( dz \) due to each variable alone:

\[ f_x(x, y) \, dx \quad \text{and} \quad f_y(x, y) \, dy \]

Diagram shows they add to get increment when both change together.

1. Use linear approximation \( L(x, y) \) when need simpler function to evaluate at some arguments \( (x, y) \) as \( f \) itself near the reference point.

2. Use differential approximation to examine how increment \( dz \) depends on increments \( dx \) and \( dy \) from reference point.

**Parametrized surface approach:**

\[ \mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle \]

- **XZ trace:** \( \mathbf{r}(x, y_0) = \langle x, y_0, f(x, y_0) \rangle \)
  \[ \mathbf{r}_x = \langle 1, 0, f_x \rangle \]

- **YZ trace:** \( \mathbf{r}(x_0, y) = \langle x_0, y, f(x_0, y) \rangle \)
  \[ \mathbf{r}_y = \langle 0, 1, f_y \rangle \]

- **Normal:** \( \mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y = \langle f_x, f_y, 1 \rangle \)

- **At \((x, y_0):**
  \[ \mathbf{n} = \langle -f_x(x, y_0), -f_y(x, y_0), 1 \rangle \]

- **At \((x_0, y):**
  \[ \mathbf{n} = \langle 0, f_y(x_0, y) \rangle \]

\[ 0 = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = -f_x(x, y_0)(x-x_0) - f_y(x_0, y_0)(y-y_0) + 1(z - f(x, y_0)) \]

or

\[ \mathbf{z} = f(x, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \]

**Real parametrized surface:**

\[ \mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle \quad \text{at} \ (\theta, z) = (\frac{\pi}{2}, 1) \]

- \( \mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle \)

- \( \mathbf{r}_z = \frac{\partial \mathbf{r}}{\partial z} = \langle 0, 0, 1 \rangle \)

\[ \mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_z = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle \]

\[ \mathbf{n} = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle \]

\[ \mathbf{r}_\theta = \langle 2 \cos \theta, 2 \sin \theta, 1 \rangle = \langle \frac{x}{2}, \frac{y}{2}, 1 \rangle \]

\[ \mathbf{n} = \langle 0, -2 \cos \theta, 2 \sin \theta \rangle \]

\[ 0 = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \]

\[ \mathbf{r}_\theta + (y-L) = 0 \]
tangent plane to level surface of 3D function

\[ f(x, y, z) = f(x_0, y_0, z_0) \] describes the level surface of \( f \) passing thru the point \( P(x_0, y_0, z_0) \).
\[ \nabla f(x, y, z) \] is perpendicular to its tangent plane so any (convenient) multiple of it can be taken as the normal vector to write the equation of this plane:

\[ \vec{n} = \nabla f(x_0, y_0, z_0) = <f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)> \]
\[ \vec{n} \cdot (x - x_0) = 0 \]

\[ f(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0 \]

We can also use the linear approximation to arrive at the same result.

The level surface of the linear approximation \( L(x, y, z) \) at \((x_0, y_0, z_0)\) passing through \((x_0, y_0, z_0)\) is the tangent plane:

\[ L(x_0, y_0, z_0) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \]

Note: \( L(x, y, z) = f(x, y, z) \).

Level surface:

\[ L(x, y, z) = L(x_0, y_0, z_0) \] becomes:

\[ f(x_0, y_0, z_0)(x - x_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = f(x, y, z) \]

\[ f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0 \]