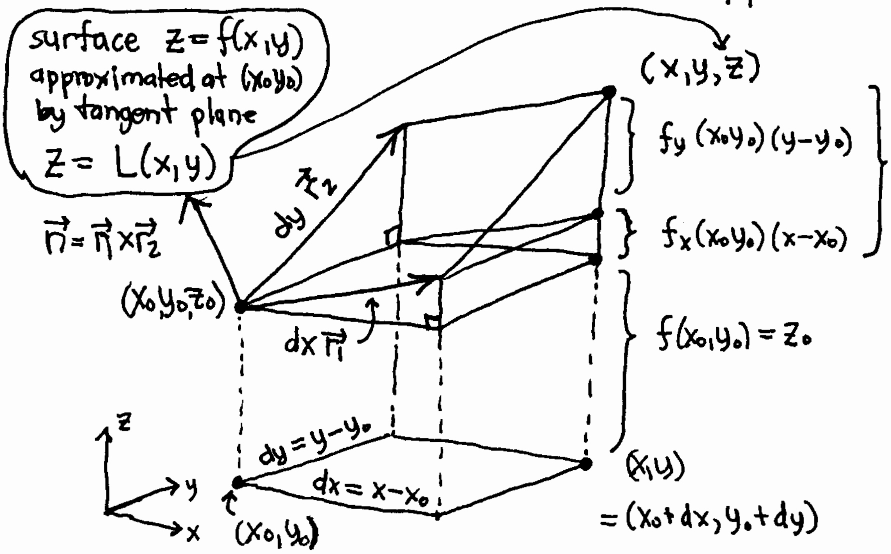
the tangent plane and the linear approximation and differentials



$$\left.\begin{aligned} &df(x_0,y_0)\\ &= f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\\ &= f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy\\ &= dz \end{aligned}\right\}\qquad \begin{aligned} z &= z_0+dz\\ &= f(x_0,y_0) + df(x_0,y_0)\\ &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\\ &= L(x,y)\end{aligned}$$

"add independent differentials"

Increments in z due to each variable alone : $f_x(x_0,y_0)\,dx$ and $f_y(x_0,y_0)\,dy$.
Diagram shows they add to get increment when both change together.

Use linear approximation $L(x,y)$ when need simpler function to evaluate at <u>same</u> arguments (x,y) as f itself, near the reference point.

Use differential approximation to examine how increment dz depends on increments dx and dy from reference point.

<u>parametrized surface approach</u>: $\vec r(x,y) = \langle x, y, f(x,y)\rangle$

x-z trace: $\vec r(x,y_0) = \langle x, y_0, f(x,y_0)\rangle \to \vec r_1 = \dfrac{\partial \vec r}{\partial x} = \langle 1, 0, f_x\rangle$

y-z trace: $\vec r(x_0,y) = \langle x_0, y, f(x_0,y)\rangle \quad \vec r_2 = \dfrac{\partial \vec r}{\partial y} = \langle 0, 1, f_y\rangle$

} tangents

normal: $\vec n = \vec r_1 \times \vec r_2 = \begin{vmatrix} i & j & k\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix} = \langle -f_x, -f_y, 1\rangle$

at (x_0,y_0): $\vec n = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1\rangle$

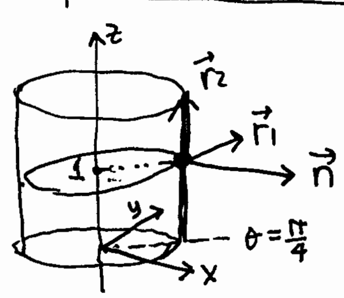
$\vec r_0 = \langle x_0, y_0, f(x_0,y_0)\rangle$

$$0 = \vec n\cdot(\vec r - \vec r_0) = -f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + 1\,(z - f(x_0,y_0))$$

or

$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

<u>real parametrized surface</u>: $\vec r(\theta,z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$ at $(\theta,z) = (\frac{\pi}{4},1)$



$\vec r_1 = \dfrac{\partial \vec r}{\partial \theta} = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

$\vec r_2 = \dfrac{\partial \vec r}{\partial z} = \langle 0, 0, 1\rangle$

$\vec n = \vec r_1 \times \vec r_2 = \begin{vmatrix} i & j & k\\ -2s & 2c & 0\\ 0 & 0 & 1\end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0\rangle$

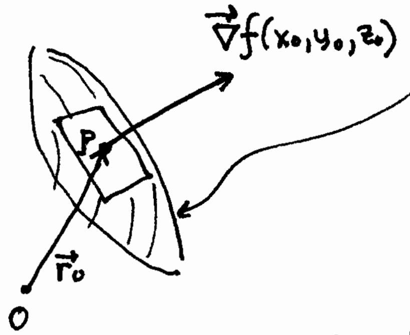
$\vec r_0 = \langle 2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}, 1\rangle = \langle \frac{2}{\sqrt2}, \frac{2}{\sqrt2}, 1\rangle$

$\vec n = \langle 2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}, 1\rangle = \langle \frac{2}{\sqrt2}, \frac{2}{\sqrt2}, 0\rangle$

$0 = \vec n\cdot(\vec r - \vec r_0) = \ldots$

$(x-\sqrt2) + (y-\sqrt2) = 0$

tangent plane to level surface of 3D function



$f(x,y,z) = f(x_0,y_0,z_0)$ describes the level surface of f passing thru the point $P(x_0,y_0,z_0)$

$\vec{\nabla} f(x_0,y_0,z_0)$ is perpendicular to its tangent plane so any (convenient) multiple of it can be taken as the normal vector to write the equation of this plane:

$$\vec{n} = \vec{\nabla} f(x_0,y_0,z_0) = \langle f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0)\rangle$$

$$\vec{r}_0 = \langle x_0,y_0,z_0\rangle$$

$$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0 \longrightarrow \langle f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0)\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0}$$

We can also use the linear approximation to arrive at the same result.

The level surface of the linear approximation $L(x,y,z)$ at (x_0,y_0,z_0) passing through (x_0,y_0,z_0) is the tangent plane:

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

Note: $L(x_0,y_0,z_0) = f(x_0,y_0,z_0)$.

Level surface:

$L(x,y,z) = L(x_0,y_0,z_0)$ becomes:

$$\cancel{f(x_0,y_0,z_0)} + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = \cancel{f(x_0,y_0,z_0)}$$

$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0 \quad ✓$$