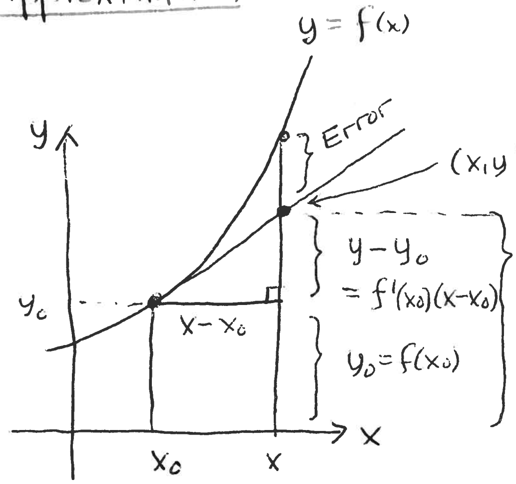
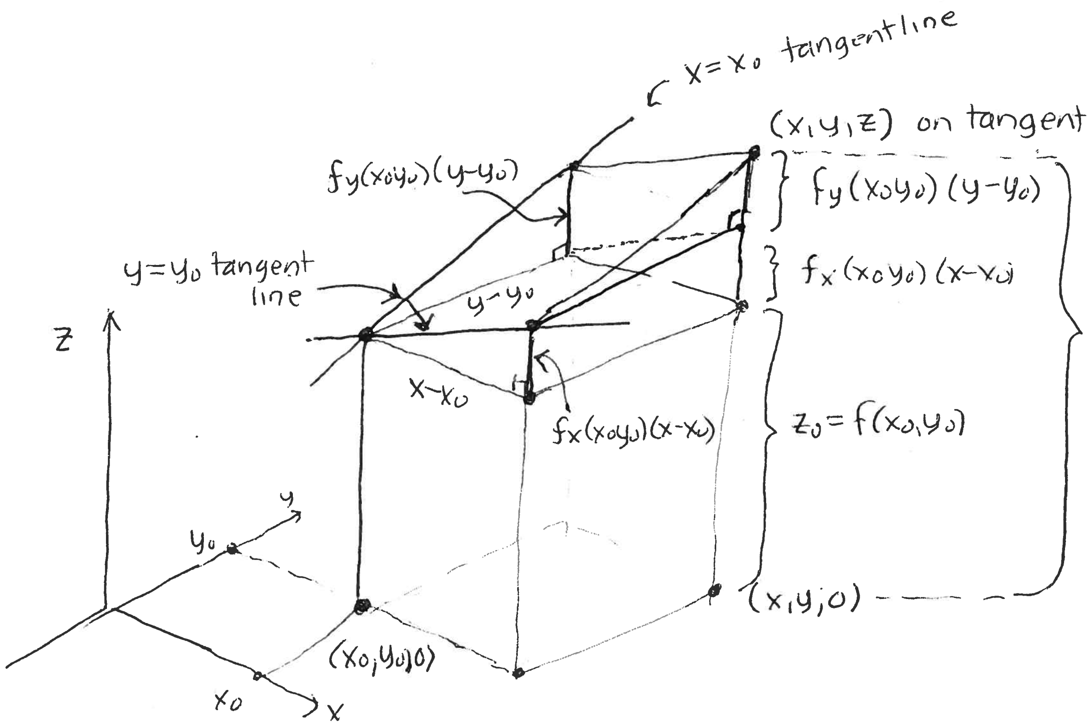
Linear Approximation

"rise" = "slope" × "run"
$y - y_0 = f'(x_0)\,(x - x_0)$



$\underbrace{y = f(x_0) + f'(x_0)(x - x_0)}_{\text{eqn of tan line}} \quad \equiv \underbrace{L(x)}_{\text{definition of linear approximation at } x = x_0}$



eqn of tan plane:

$$z = f(x_0, y_0) + f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0)$$

$\equiv \underbrace{L(x, y)}$ definition of linear approximation at (x_0, y_0)

(stack triangles at opposite sides on top of rectangles)

Complete $y = y_0$ cross-section tangent line for interval from x_0 to x plus $x = x_0$ cross-section tangent line from y_0 to y to a parallelogram — defines the tangent plane containing both tangent lines — and its far corner is the point (x, y, z) satisfying the tangent plane equation

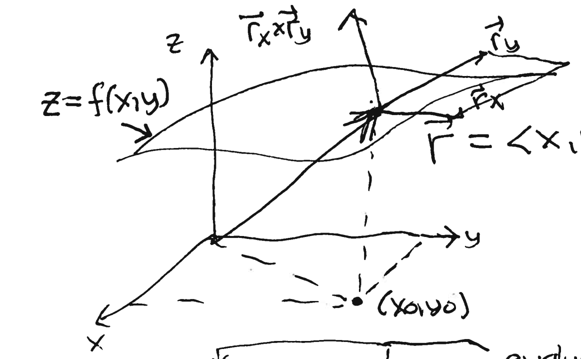
<u>Independent</u> Calc1 increments to function add —

for $f(x, y, z)$:

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0 y_0 z_0)(x - x_0) + f_y(x_0 y_0 z_0)(y - y_0) + f_z(x_0 y_0 z_0)(z - z_0)$$

etc

tangent plane to surface graph in space



$\vec r = \langle x, y, f(x,y)\rangle$

tangents to coord grid lines on surface:

$\vec r_x = \langle 1, 0, f_x(x,y)\rangle$

$\vec r_y = \langle 0, 1, f_y(x,y)\rangle$

$\vec n = \vec r_x \times \vec r_y = \dots \langle -f_x(x,y), -f_y(x,y), 1\rangle$

upward normal

evaluate at $\vec r(x_0,y_0) = \langle x_0, y_0, f(x_0,y_0)\rangle$

$$0 = \vec n_0 \cdot (\vec r - \vec r_0)$$
$$= \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1\rangle \cdot \langle x-x_0, y-y_0, z-f(x_0,y_0)\rangle$$
$$= -f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + z - f(x_0,y_0)$$

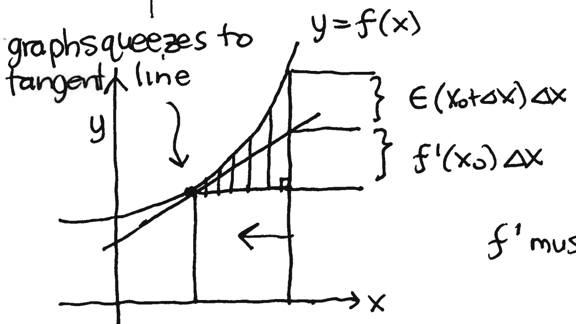
$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

linear approximation to f at (x_0,y_0)

requires

if f_x, f_y continuous at (x_0,y_0)
f is differentiable there and
tangent plane exists & approximates the graph nearby
& tilts continuously to nearby points

independent variable increments add to value at (x_0,y_0)

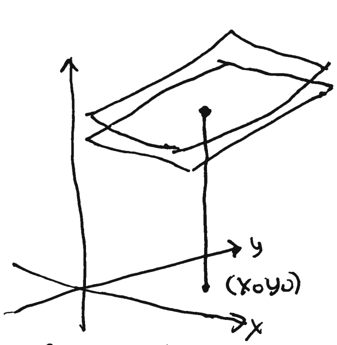


differentiability at $x=x_0$

fractional error goes to 0:

$$\frac{\epsilon(x_0+\Delta x)\Delta x}{f'(x_0)\,\Delta x} \to 0 \quad \text{so} \quad \lim_{\Delta x\to 0} \epsilon(x_0+\Delta x) = 0$$

f' must be continuous at x_0 if can "roll" tangent line along graph nearby (varying x_0 away from its value)



differentiability at $(x,y)=(x_0,y_0)$

$$\text{error} = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$
$$= \underbrace{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y}_{\text{linear approx to increment}} + \underbrace{\epsilon_1(x_0+\Delta x, y_0+\Delta y)\Delta x + \epsilon_2(x_0+\Delta x, y_0+\Delta y)\Delta y}_{\text{error}}$$

$$\text{fractional error} = \frac{\epsilon_1(x_0+\Delta x, y_0+\Delta y)\Delta x + \epsilon_2(x_0+\Delta x, y_0+\Delta y)\Delta y}{f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y} \to 0$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left(\text{fractional error}\right) = 0$$ requires same for $\epsilon_1(x_0+\Delta x, y_0+\Delta y)$ & $\epsilon_2(x_0+\Delta x, y_0+\Delta y)$