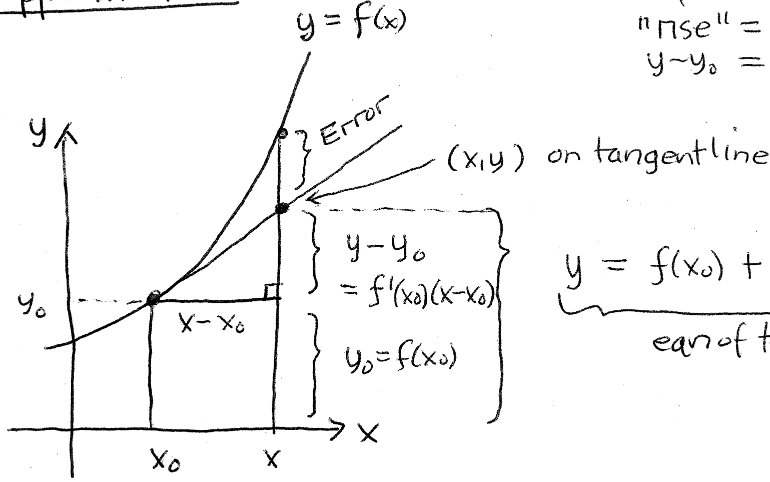
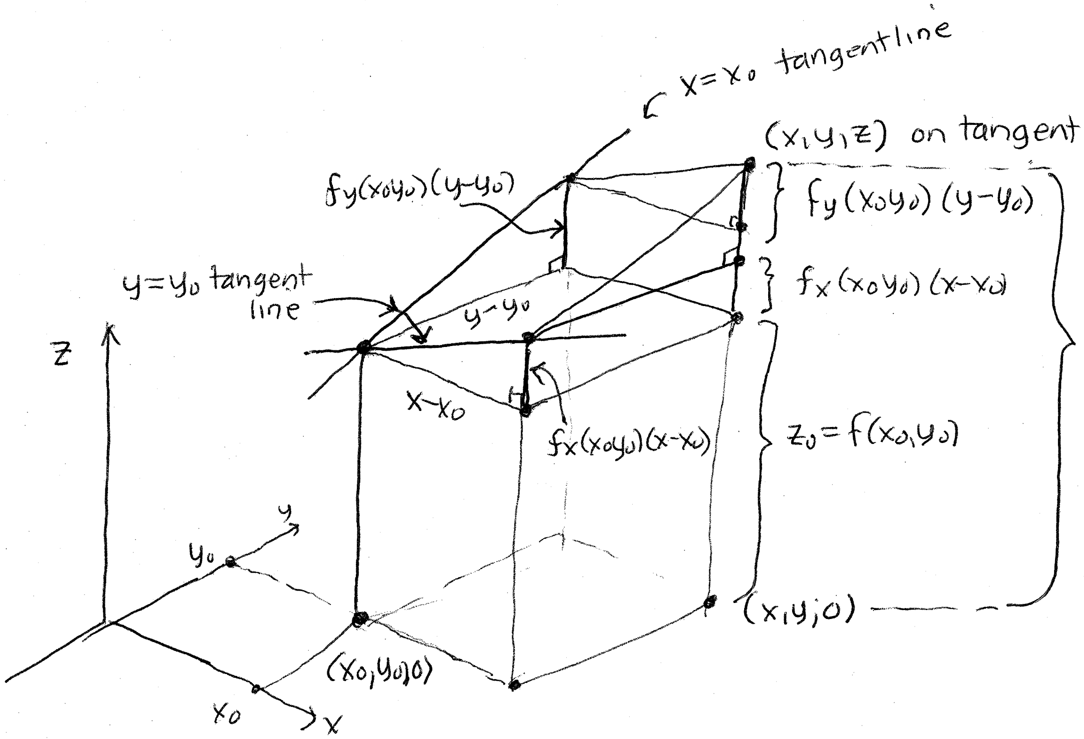
Linear Approximation



"rise" = "slope" × "run"

$y - y_0 = f'(x_0)\ (x - x_0)$

$$\underbrace{y = f(x_0) + f'(x_0)(x - x_0)}_{\text{eqn of tan line}} \equiv \underbrace{L(x)}_{\text{definition of linear approximation at } x = x_0}$$



eqn of tan plane:

$$z = f(x_0, y_0) + f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0)$$

$$\equiv \underbrace{L(x, y)}_{\text{definition of linear approximation at } (x_0, y_0)}$$

(stack triangles at opposite sides on top of rectangles)

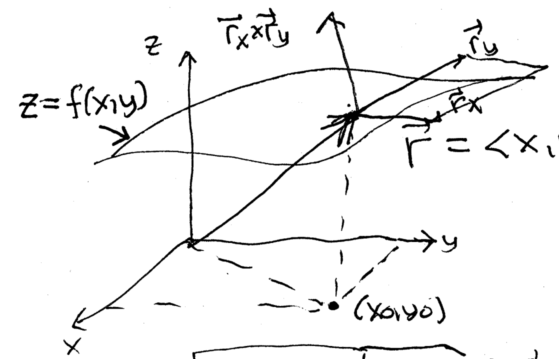
Complete $y = y_0$ cross-section tangent line for interval from x_0 to x plus $x = y_0$ cross-section tangent line from y_0 to y to a parallelogram — defines the tangent plane containing both tangent lines — and its far corner is the point (x, y, z) satisfying the tangent plane equation

Independent Calc1 increments to function add — for $f(x y z)$:

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0 y_0 z_0)(x - x_0) + f_y(x_0 y_0 z_0)(y - y_0) + f_z(x_0 y_0 z_0)(z - z_0)$$

etc.

tangent plane to surface graph in space



$\vec{r} = \langle x, y, f(x,y) \rangle$

tangents to coord grid lines on surface:

$\vec{r}_x = \langle 1, 0, f_x(x,y) \rangle$

$\vec{r}_y = \langle 0, 1, f_y(x,y) \rangle$

$\vec{n} = \vec{r}_x \times \vec{r}_y = \dots \langle -f_x(x,y), -f_y(x,y), 1 \rangle$

upward normal

evaluate at $\vec{r}(x_0,y_0) = \langle x_0, y_0, f(x_0,y_0) \rangle$

$$0 = \vec{n}_0 \cdot (\vec{r} - \vec{r}_0)$$
$$= \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle \cdot \langle x - x_0, y - y_0, z - f(x_0,y_0) \rangle$$
$$= -f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + z - f(x_0,y_0)$$

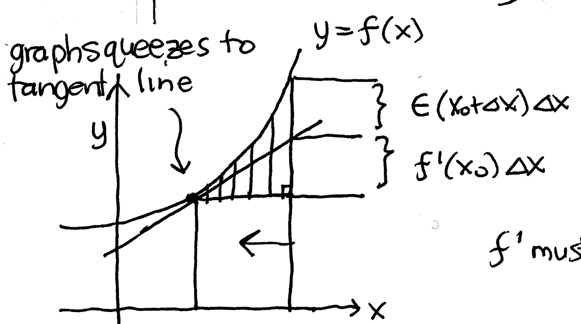
$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

linear approximation to f at (x_0,y_0)

requires

if f_x, f_y continuous at (x_0,y_0)
f is differentiable there and
tangent plane exists & approximates the graph nearby
& tilts continuously to nearby points

independent variable increments add to value at (x_0,y_0)

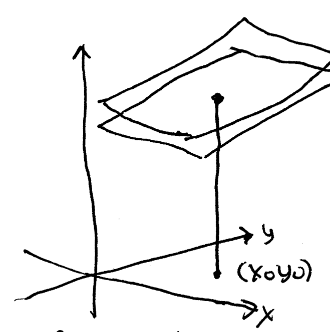


differentiability at $x = x_0$

fractional error goes to 0:

$$\frac{\epsilon(x_0+\Delta x)\Delta x}{f'(x_0)\Delta x} \to 0 \quad \text{so} \quad \lim_{\Delta x \to 0} \epsilon(x_0+\Delta x) = 0$$

f' must be continuous at x_0 if can "roll" tangent line along graph nearby (varying x_0 away from its value)



differentiability at $(x,y) = (x_0,y_0)$

$$\text{error} = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$
$$= \underbrace{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y}_{\text{linear approx to increment}} + \underbrace{\epsilon_1(x_0+\Delta x, y_0+\Delta y)\Delta x + \epsilon_2(x_0+\Delta x, y_0+\Delta y)\Delta y}_{\text{error}}$$

$$\text{fractional error} = \frac{\epsilon_1(x_0+\Delta x, y_0+\Delta y)\Delta x + \epsilon_2(x_0+\Delta x, y_0+\Delta y)\Delta y}{f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y} \to 0$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left(\text{fractional error}\right) = 0$$ requires same for $\epsilon_1(x_0+\Delta x, y_0+\Delta y)$ & $\epsilon_2(x_0+\Delta x, y_0+\Delta y)$

Differentials

If we are only interested in the linear approximation CHANGE in the function value, then only the sum of the linear increments for each independent variable are needed:

$$\underbrace{f(x,y)}_{z} \approx \underbrace{f(x_0,y_0)}_{z_0} + f_x(x_0,y_0)\underbrace{(x-x_0)}_{dx} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{dy} \qquad \text{linear approximation}$$

} convenient notation for increments. dx, dy, dz

$$\underbrace{z-z_0}_{dz} = f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0)$$

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0) \qquad \equiv df\underbrace{(x_0,y_0,dx,dy)}_{\text{function of 4 independent variables}}$$

Now no longer need to use subscript "o":

$$dz = f_x(x,y)dx + f_y(x,y)dy = df(x,y) \quad \leftarrow \text{appropriate in function notation}$$

$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

appropriate if no "named function"

example 1. Area A of rectangle of sides x and y: $A = xy$ $\left< \begin{array}{l} \frac{\partial A}{\partial x} = y \\ \frac{\partial A}{\partial y} = x \end{array}\right.$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = y\,dx + x\,dy$$

If sides change from x, y to $x+dx$, $y+dy$ then A changes from A to $A+dA$ (approximately), and the fractional change is $\frac{dA}{A} = \frac{y\,dx + x\,dy}{xy} = \frac{dx}{x} + \frac{dy}{y}$ namely the sum of the fractional changes in the dimensions.

example 2 The equivalent 4in × 6in photo print format in Europe with the same dimension ratio $\frac{10}{15} = \frac{2}{3} = \frac{4}{6}$ is the 10cm × 15cm format.

If an actual print is measured with a millimeter ruler and found to have the dimensions 10 cm × 15 cm to within an error of ± 0.2 mm = ± 0.02 cm, what is the computed error in their ratio? What is the percentage error?

$$R = \frac{x}{y} \qquad x = 10,\ |dx| \le 0.02 \qquad y = 15,\ |dy| \le 0.02$$

$$\frac{\partial R}{\partial x} = \frac{1}{y}, \quad \frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(xy^{-1}) = -\frac{x}{y^2}, \quad dR = \frac{1}{y}dx - \frac{x\,dx}{y^2} = \frac{y\,dx - x\,dy}{y^2}$$

$$|dR| = \frac{|y\,dx - x\,dy|}{y^2} \le \frac{y|dx| + x|dy|}{y^2} \qquad \leftarrow |A+B| \le |A| + |B| \,!$$

$$= \frac{15(0.2) + 10(0.2)}{15^2} = 0.2\left(\frac{25}{15^2}\right) = 0.2\left(\frac{5^2}{3^2 5^2}\right) = \frac{0.2}{9} \approx 0.022$$

The error in the ratio is about 0.022 (only 1 significant figure is warranted).

$$\frac{|dR|}{R} \le \frac{\frac{0.2}{9}}{2/3} = \frac{0.2}{6} = \frac{0.1}{3} \approx 0.033 \qquad \text{about 3\% error.}$$

HW problem Calculate (using differentials) the absolute and percentage difference in the area of the exact 4in × 6in print format relative to the 10 cm × 15 cm format. Compare them to the exact differences not using differentials.