S14.5.49

$$\begin{cases} x = r\cos\theta & \dfrac{\partial x}{\partial r} = \cos\theta, \quad \dfrac{\partial x}{\partial \theta} = -r\sin\theta \qquad y = r\sin\theta \quad \dfrac{\partial y}{\partial r} = \sin\theta, \quad \dfrac{\partial y}{\partial \theta} = r\cos\theta \\ z = z(x,y) \end{cases}$$

oops ⟨first error⟩

- $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\underbrace{\frac{\partial x}{\partial r}}_{\cos\theta} + \frac{\partial z}{\partial y}\underbrace{\frac{\partial y}{\partial r}}_{\sin\theta} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$

- $$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$ (θ held constant)

$$= \cos\theta\left[\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}_{\frac{\partial^2 z}{\partial x^2}}\underbrace{\frac{\partial x}{\partial r}}_{\cos\theta} + \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)}_{\frac{\partial^2 z}{\partial y \partial x}}\underbrace{\frac{\partial y}{\partial r}}_{\sin\theta}\right] + \sin\theta\left[\text{⟨oops⟩} \quad \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\right]$$

I copied wrong $\frac{\partial x}{\partial \theta}$ for $\frac{\partial y}{\partial r}$....

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}_{\frac{\partial^2 z}{\partial x \partial y}}\underbrace{\frac{\partial x}{\partial r}}_{\cos\theta} + \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)}_{\frac{\partial^2 z}{\partial y^2}}\underbrace{\frac{\partial y}{\partial r}}_{\sin\theta}$$

$$= \cos^2\theta\frac{\partial^2 z}{\partial x^2} + \cos\theta\sin\theta\frac{\partial^2 z}{\partial y \partial x} \quad \text{⟨oops⟩} \quad + \cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}$$

order doesn't matter $\frac{\partial^2 z}{\partial x \partial y}$

- $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta} + \frac{\partial z}{\partial y}\underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

- $$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}\right) = -r\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial z}{\partial x}\right) + r\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial z}{\partial y}\right)$$

$$= -r\left[\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)\right] + r\left[-\sin\theta\frac{\partial z}{\partial y} + \cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)\right]$$

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)}_{\frac{\partial^2 z}{\partial x^2}}\underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta} + \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)}_{\frac{\partial^2 z}{\partial y \partial x}}\underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta} \qquad \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}_{\frac{\partial^2 z}{\partial x \partial y}}\underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta} + \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)}_{\frac{\partial^2 z}{\partial y^2}}\underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta}$$

$$= -r\left[\cos\theta\frac{\partial z}{\partial x} - r\sin^2\theta\frac{\partial^2 z}{\partial x^2} + r\cos\theta\sin\theta\frac{\partial^2 z}{\partial y \partial x}\right]$$
$$+ r\left[-\sin\theta\frac{\partial z}{\partial y} - r\cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y} + r\cos^2\theta\frac{\partial^2 z}{\partial y^2}\right]$$

$= \frac{\partial^2 z}{\partial x \partial y}$ order doesn't matter

$$= -r\cos\theta\frac{\partial z}{\partial x} - r\sin\theta\frac{\partial z}{\partial y} + r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2} + r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2} - 2r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y}$$

(oops, almost copied wrong again)

- $$\begin{aligned} &\frac{\partial^2 z}{\partial r^2} && = \cos^2\theta\frac{\partial^2 z}{\partial x^2} + \sin^2\theta\frac{\partial^2 z}{\partial y^2} + 2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y} \\ &+\frac{1}{r}\frac{\partial z}{\partial r} && + \frac{1}{r}\cos\theta\frac{\partial z}{\partial x} + \frac{1}{r}\sin\theta\frac{\partial z}{\partial y} \\ &+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} && + \sin^2\theta\frac{\partial^2 z}{\partial x^2} + \cos^2\theta\frac{\partial^2 z}{\partial y^2} - 2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y} - \frac{1}{r}\cos\theta\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial y} \end{aligned}$$

$$= 1\cdot\frac{\partial^2 z}{\partial x^2} + 1\cdot\frac{\partial^2 z}{\partial y^2} + 0\frac{\partial^2 z}{\partial x \partial y} + 0\frac{\partial z}{\partial x} + 0\frac{\partial z}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

Whew! A few missteps along the way but I managed to catch them & recover.